THE TWELFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Twelfth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Williams College, Williamstown, Mass., on Thursday and Friday, September 7–8, 1905. The following twenty-eight members of the Society were in attendance:

Professor G. A. Bliss, Professor W. G. Bullard, Dr. W. H. Bussey, Professor A. S. Chessin, Professor F. N. Cole, Professor L. L. Conant, Professor D. R. Curtiss, Professor E. W. Davis, Professor W. P. Durfee, Professor F. C. Ferry, Professor W. B. Fite, Professor A. S. Gale, Professor J. G. Hardy, Professor E. V. Huntington, Professor J. I. Hutchinson, Dr. Edward Kasner, Professor Frank Morley, Professor E. D. Roe, Dr. F. H. Safford, Miss M. E. Sinclair, Dr. C. H. Sisam, Professor Virgil Snyder, Professor W. E. Story, Professor H. W. Tyler, Professor H. S. White, Professor T. W. D. Worthen, Professor J. W. A. Young, Professor J. W. Young.

Two sessions were held on Thursday, and a third session on Friday morning. Professor Frank Morley and Professor F. C. Ferry were elected chairmen. The Council announced the election of the following persons to membership in the Society: Lieutenant Colonel C. P. Echols, U. S. Military Academy; Professor G. B. Guccia, University of Palermo; Professor H. B. Evans, University of Pennsylvania; Dr. A. M. Hiltebeitel, Princeton University; Dr. J. M. Poor, Dartmouth College; Professor J. E. Williams, Virginia Polytechnic Institute. Eight applications for membership in the Society were received.

It was decided to hold the annual meeting of the Society in New York on Thursday and Friday, December 28–29. A committee, consisting of Professors E. H. Moore, James Pierpont, and G. A. Miller, was appointed to prepare and report at the October meeting a list of nominations of officers and other members of the Council to be voted for at the annual meeting.

At the close of the Thursday morning session the members were conducted through the grounds and buildings of Williams College and the collection of mathematical models was shown. On Friday afternoon the members assembled at the house of

President Hopkins and through the courtesy of the college were taken in carriages on an excursion into the Berlin mountains whose less accessible regions were traversed on foot. Several foot tours were also made on Saturday. The hospitality of the college authorities was appropriately recognized by appreciative resolutions at the close of the meeting.

The following papers were read at the meeting:

(1) Dr. W. H. BUSSEY: "Galois field tables for $p^n \leqq 169$."

(2) Dr. EDWARD KASNER: "A geometric property of the trajectories of dynamics."

(3) Professor G. A. BLISS: "A generalization of the notion of angle."

(4) Professor W. B. FITE: "Irreducible linear homogeneous groups."

(5) Dr. SAUL EPSTEEN: "Note on the structure of hypercomplex number systems."

(6) M. MAURICE FRÉCHET: "Sur l'écart de deux courbes et sur les courbes limites."

(7) Mr. RICHARD MORRIS: "On the expressibility of the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$ in terms of theta series."

(8) Professor J. I. HUTCHINSON: "On certain hyperabelian functions which are expressible by theta series."

(9) Mr. N. J. LENNES: "Concerning real functions of one real variable which are completely determined over an interval by the values of the function and its derivatives for one value of the independent variable."

(10) Dr. W. A. MANNING: "On the arithmetic nature of the coefficients in groups of finite monomial linear substitutions."

(11) Dr. MAX MASON: "On the boundary value problems of linear ordinary differential equations of the second order."

(12) Professor G. A. MILLER: "On the possible number of operators of order 2 in a group of order 2^m."

(13) Professor FRANK MORLEY: "On two cubic curves in triangular relation."

(14) Dr. C. H. SISAM: "On the determination of the nodal curve on a ruled surface."

(15) Professor A. S. CHESSIN: "On the strains and stresses in a rapidly revolving circular disc."

(16) Professor L. E. DICKSON: "On the quaternary linear homogeneous groups modulo p of order a multiple of p."

(17) Professor L. E. DICKSON: "On finite algebras."

(18) Professor VIRGIL SNYDER: "On a type of rational twisted curves."

(19) Professor E. J. TOWNSEND: "Arzelà's condition for the continuity of a function defined by a series of continuous functions."

(20) Professor H. S. WHITE: "Rational plane curves as related to Riemann transformations."

(21) Professor F. R. MOULTON: "A class of periodic solutions of the problem of three bodies."

(22) Dr. C. N. HASKINS: "Note on the differential invariants of a surface and of space."

(23) Professor E. V. HUNTINGTON: "The continuum as a type of order: an exposition of the modern theory."

M. Fréchet's paper was communicated to the Society through Professor E. H. Moore, Mr. Morris's paper through Professor Hutchinson. In the absence of the authors, Mr. Morris's paper was read by Professor Hutchinson, Dr. Haskins's paper by Professor Bliss, and the papers of Dr. Epsteen, M. Fréchet, Mr. Lennes, Dr. Manning, Dr. Mason, Professor Miller, Professor Dickson, Professor Townsend, and Professor Moulton were read by title.

The papers of Dr. Bussey and Professor Townsend appeared in the October BULLETIN. Those of Dr. Kasner, Dr. Epsteen, Dr. Manning, and Professor Miller are included in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

3. Geometry has to do with a set of elements called points, which is divided into subsets called lines. If it is postulated that the elements of the original set can be put into one to one correspondence with the real number ratios $x : y : z$, then the points having ratios of the form $x : y : 1$ can be represented geometrically in the ordinary cartesian plane. A curve may be defined as the set of points corresponding to two equations $x = \phi(t)$, $y = \psi(t)$, and the length of a curve as the value of a definite integral of the form

$$I = \int F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt.$$

The nature of the geometry depends then upon the integral chosen to represent length. Hamel has used this notion in his determination of all geometries in which straight lines are the shortest distances, but without giving a corresponding generalization of angle. In Professor Bliss's paper such a generalization is developed, together with its application to surface theory and the so-called non-euclidean geometries.

4. It is well known that the number of variables in an irreducible linear homogeneous group is a divisor of the order of the group. In Professor Fite's paper this number is determined more closely for certain groups, and is definitely fixed for metabelian groups. A necessary and sufficient condition that a group whose order is a power of a prime or a group that is the direct product of such groups be simply isomorphic with an irreducible group is derived. These results for linear groups lead to certain properties of abstract groups. Finally the class of an irreducible group of order p^m in p variables (p a prime) is determined. This paper forms part of an article that is to appear in the *Transactions* under the title "Groups whose orders are powers of a prime."

6. The chief result of M. Fréchet's paper is to define a number called interspace (écart) which has the same properties in the theory of curves as distance in the theory of points: 1°. Two continuous curves C_1, C_2 being given, their interspace is a number, $(C_1, C_2) \geqq 0$, uniquely determined. 2°. The necessary and sufficient condition that two continuous curves coincide is that their interspace vanish. 3°. Given any three continuous curves C_1, C_2, C_3, then $(C_1, C_2) \leqq (C_1, C_3) + (C_3, C_2)$. Finally the concept of interspace is introduced into the theory of limiting curves.

7. Mr. Morris has studied the effect of a monodromy of the branch points of the Riemann surface $w^\nu = (z-a)^\alpha(z-b)^\beta (z-c)^\gamma(z-d)^\delta$, where $\alpha + \beta + \gamma + \delta \equiv 0 \pmod{\nu}$, on the moduli of periodicity of the integrals of the first kind. If the surface be of genus p, and if $2n$ denote the number of linearly independent integrals which are not of lower genus, it is found that the number of linearly independent parameters in terms of which the moduli of these integrals are linearly expressible is in general not less than n. Under special condi-

tions on a, β, γ, δ the minimum possible value for this number reduces to $n/2$, or $n/4$. An important consequence is that the automorphic functions of class $(0, 3; l_1, l_2, l_3)$ cannot be expressed in terms of theta series containing only one variable in any cases other than those already known.

8. In the paper by Professor Hutchinson it is shown how the moduli of the theta functions can be specialized so as to be linearly expressible in terms of a certain number of independent parameters $\zeta_1, \zeta_2, \cdots$. Certain groups of transformations on these parameters correspond to linear transformations of the theta functions and thus lead to hyperabelian functions of $\zeta_1, \zeta_2, \cdots$, which are expressible in terms of theta series.

9. In Mr. Lennes's paper the following theorem is proved:

If all the derivatives of a function $f(x)$ exist for every point of a segment $a \cdots b$ then a necessary and sufficient condition that $f(x)$ shall be expressible by means of Taylor's series at every point of the segment $a \cdots b$ is that every point x_k of this segment shall lie on a segment σ_k such that for some fixed number $k_k \neq 0$

$$\lim_{n=\infty} \frac{f^{(n)}(x) k_k^n}{n!} = 0$$

uniformly for x on σ_k.

This condition differs essentially from that given by Pringsheim, though the proof in some respects is similar.

It is then shown that, if for a certain point the Taylor series is divergent or is convergent but fails to represent the function, then the function is not uniquely determined on any interval by the value of the function and those of its derivatives at that point, consequently it is impossible to devise an algorithm in terms of $f^{(i)}(x_0)$ $(i = 0, \cdots, \infty)$ which shall represent the function over an interval containing x_0 in case Taylor's expansion fails.

As a lemma a necessary and sufficient condition is given that a series of continuous functions whose sum is a continuous function shall be uniformly convergent.

11. Dr. Mason considered the general boundary conditions:

$$a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = A$$
$$b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = B,$$

in connection with certain ordinary linear differential equations of the second order which contain a parameter. Some of the results of the paper were published in the *Comptes rendus*, April 27, 1905.

13. Professor Morley's paper is in abstract as follows: The fixed points of a collineation belonging to a pencil, in a plane, are on a cubic curve ϕ; and the fixed lines are on a cubic curve f. Thus ϕ and f admit an infinity of Poncelet triangles. By joining every point to the corresponding point in a collineation we pick out of the ∞^3 elements of the plane a nexus of ∞^2 elements. The elements common to two nexuses form the two cubics. Each triangle forms with every element a constant double ratio. The two cubics have 6 contacts, 6 simple common points, and 6 simple common lines.

14. In Dr. Sisam's paper the equation of a plane curve in one to one correspondence with the nodal curve of a rational scroll was obtained. Let the equations of the scroll be: $x_i = a_i(u) + vb_i(u)$ $(i = 1, 2, 3, 4)$. If the generators determined by $u = u$ and $u = u'$ intersect, then the determinant $|a_1(u)a_2(u')b_3(u)b_4(u')| = 0$. This equation, after division by $(u - u')^2$, may be expressed in terms of $u + u'$ and uu' only. Call it $F(u + u', uu') = 0$. Put $u + u' = \xi$, $uu' = \eta$ and consider $F(\xi,\eta) = 0$ as the equation of a plane curve. It then follows that each irreducible component of $F(\xi,\eta) = 0$ determines an irreducible component of the nodal curve on the surface. If the latter is double only, then the number of times it is cut by an arbitrary generator equals the order, and its genus equals the genus of the component of $F = 0$ which determines it.

The components of the double developable and their properties analogous to those of the double curve are also determined by the components of $F(\xi,\eta) = 0$. The equation in u found by putting $u' = u$ in $F(u+u', uu') = 0$ determines the parameters of the torsal generators.

15. If the radial and axial displacements u and w caused by the rotation of a circular disc of thickness $2h$ and radius R, at a point (r, z), be expressed as follows:

$$(1) \qquad u = -\frac{\rho\omega^2}{8(2\mu + \lambda)} r^3 + \sum_0^\infty u_n z^{2n},$$

$$w = \quad w_n z^{2n+1}; \tag{2}$$

if, moreover, we put

$$z_n = h^{2n}\left\{(2\mu+\lambda)\frac{1}{r}\frac{dru_n}{dr} + \lambda(2n+1)w_n\right\}, \tag{3}$$

$$y^n = h^{2n}\left\{\lambda\frac{1}{r}\frac{dru_n}{dr} + (2\mu+\lambda)(2n+1)w_n\right\}, \tag{4}$$

then we have, to determine x_n and y_n, the system of differential equations

$$h^2 Dx_n = (2n+1)(2n+2)y_{n+1}, \tag{5}$$

$$h^2 Dy_n = -(2n+1)(2n+2)(x_{n+1} + 2y_{n+1}) \tag{6}$$

$(n = 0, 1, 2, \cdots)$,

where D indicates the operation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}.$$

The surface conditions for $z = \pm h$ furnish two equations, viz.:

$$\sum\frac{x_n}{2n+1} = C, \qquad \sum y_n = \frac{\lambda\rho\omega^2}{2(2\mu+\lambda)}r^2. \tag{7}$$

The integration of the system (5)–(6), together with equations (7), gives the following results:

$$(n>1)\, x_n = \sum C_{n,e}J_0\left(\frac{\lambda_e}{h}r\right), \qquad (n \geqq 1)\, y_n = \sum C'_{n,e}J_0\left(\frac{\lambda_e}{h}r\right)$$

$$x_0 = C + \sum C_{0,e}J_0\left(\frac{\lambda_e}{h}r\right),$$

$$x_1 = -\frac{\lambda\rho\omega^2}{2\mu+\lambda}h^2 + \sum C_{1,e}J_0\left(\frac{\lambda_e}{h}r\right), \tag{8}$$

$$y_0 = \frac{\lambda\rho\omega^2}{2(2\mu+\lambda)}\mu^2 + \sum C'_{0,e}J_0\left(\frac{\lambda_e}{h}r\right),$$

where the λ_s are the roots of the transcendental equation

$$\sin 2ix + 2ix = 0 \tag{9}$$

and the coefficients $C_{n,s}$, $C'_{n,s}$ are given by the formulas

$$C_{n,s} = \left(1 + \frac{2n}{1 + \cos^2 i\lambda_s}\right)\frac{\lambda_s^{2n}}{2n!}C_{0,s},$$

$$C'_{n,s} = \left(\frac{\sin^2 i\lambda_s - 2n}{1 + \cos^2 i\lambda_s}\right)\frac{\lambda_s^{2n}}{2n!}C_{0,s},$$

while the coefficients C and the $C_{0,s}$ are determined from the two remaining surface conditions for $r = R$.

Once the x_n and y_n are determined, the w_n are obtained from (3)–(4) directly and the u_n by a simple integration. Thus will be found the displacements u and w and by their means the strains and stresses at any point of the disc.

This solution of the problem does not readily lend itself to practical applications on account of its intricate form. A simplified solution will be the subject of a subsequent paper.

16. The group of all quaternary linear homogeneous substitutions of determinant unity modulo p is of order $p^6(p^4-1)(p^3-1)(p^2-1)$. It is shown by Professor Dickson that any subgroup of order a multiple of p^6 is contained in a group conjugate with one of three maximal subgroups

$$\begin{array}{ll} \text{(I)} & (a_{ij}),\ a_{13} = a_{14} = a_{23} = a_{24} = 0,\ |a| = 1; \\ \text{(II)} & (a_{ij}),\ a_{14} = a_{24} = a_{34} = 0,\ |a| = 1; \\ \text{(III)} & (a_{ij}),\ a_{12} = a_{13} = a_{14} = 0,\ |a| = 1. \end{array}$$

Here II and III are simply isomorphic groups of order $p^3(p^3-1)(p^3-p)(p^3-p^2)$, while I is of order $p^6(p^2-1)^2(p-1)$. Next, let H be a subgroup of order p^5N, N prime to p, $p > 2$. If its G_{p^5} is invariant, H is contained in (a_{ij}), $a_{ij} = 0 (j > i)$ except a certain $a_{s-1,s}$. If G_{p^5} is not invariant, H is conjugate with one of the types: (I) with the $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ defining a binary group of order prime to p; (I) with the same restriction on the $\left(\begin{smallmatrix} a_{33} & a_{34} \\ a_{34} & a_{44} \end{smallmatrix}\right)$; (I) with

$$a_{11} = \pm\, \epsilon^{-1}a_{33},\ a_{12} = \pm\, \mu^{-1}\epsilon^{-1}a_{34},\ a_{21} = \pm\, \mu\epsilon^{-1}a_{43},\ a_{22} = \pm\, \epsilon^{-1}a_{44},$$

where $e = a_{33}a_{44} - a_{34}a_{43}$, and μ is a fixed integer, either 1 or a particular quadratic non-residue of p. According to the restrictions placed on the sign $\pm$ and on e, the last group is of order $p^5(p^2-1)t$, t any chosen factor of $2(p-1)$. The paper has been offered to the *American Journal of Mathematics*.

17. Of the algebras in which all but one of the postulates of a field hold (*Transactions*, 1905, page 201), the most interesting are those in which the commutativity or associativity of multiplication is not assumed. In his second paper, Professor Dickson treats at length finite algebras whose elements form a group under addition, whose elements $\neq 0$ form a group under multiplication, and for which the left-hand distributive law holds. (If also the right-hand distributive law holds, the algebra is a field.) It is shown that the elements may be given the notation $(a_1, \cdots, a_n)$, each a_i an integer taken modulo p a prime, and that $(a_i) + (b_i) = (a_i + b_i)$. The algebra is a field if $p^n - 1$ and n are relatively prime, or if $n = 1$. In the remaining cases, non-field algebras exist; a typical example is

$$(a, b) \times (x, y) = (ax + evby,\ bx + eay),$$

where v is a fixed quadratic non-residue of p, and $e = \pm 1$ according as $a^2 = vb^2$ is a quadratic residue or non-residue of p.

In the second type of algebra, the elements form a group under addition, the product of any two is in the set, the right-hand identity and inverse occur, and multiplication is commutative and distributive; the omitted postulate is associativity of multiplication. If the algebra is finite the elements are $(a_1, \cdots, a_n)$, the a_i taken modulo p, prime, and $(a_i) + (b_i) = (a_i + b_i)$. If $n < 3$, the algebra is a field. Non-field algebras are found for n any multiple of 3. The remaining cases were not examined. If F is any finite or infinite field not having modulus 2, an algebra having all the required properties is that with the units $1, i, j$, and coördinates in F, where $i^2 = j$, $ij = ji = b + \beta i$, $j^2 = -\beta^2 = 8bi - 2\beta j$, where b and β are such that $x^3 = b + \beta x$ is irreducible in F. In this linear, non-associative algebra, every element has an unique inverse. When F is the field of integers modulo p, this algebra is the only such non-field algebra for $p = 3$, 5, 7 or 11, the cases examined exhaustively.

18. The curves discussed by Professor Snyder are rational, of order n, and have two tangents of $n-1$ pointic contact at

A and B. The osculating plane at any point P cuts the curve in $n-3$ points Q_i. The surface generated by the lines PQ_i is composite, each factor being rational and of order $2n-4$. The given curve is tacnodal on each such surface, and the residual curve breaks up into $n-2$ factors, each being cut twice by every generator. If n is even, one of the points Q_k is the harmonic conjugate of P as to A and B. The lines PQ_k generate a scroll having AB for multiple directrix, upon which the given curve is an asymptotic line.

20. In the geometry of birational transformations in a plane or other rational surface, the Cremona group is a subgroup of the general algebraic or Riemann group, hence the former may have some invariants distinct from those of the latter. Professor White points out the fact that this is not so when the object to which the transformations are applied is a single rational curve on the surface.

21. Professor Moulton's paper discusses the existence, properties, and construction of certain periodic solutions of the problem of three bodies in which the orbits reduce to circles with the vanishing of certain parameters. With proper specialization of the parameters the problem is that of the periodic orbits of two mutually disturbing finite planets; with another specialization, the problem of the periodic solution of a finite satellite disturbed by the sun; and with certain further limitations, the moon's variational orbit. The results are developed literally so as to apply to any case in the various types, and numerical illustrations are exhibited. They include the moon's variational and parallactic orbit carried out to an order of accuracy which leaves nothing to be desired, and two classes of the periodic orbits discovered by Darwin (*Acta Mathematica*, volume 21) from numerical experiments.

22. Dr. Haskins's paper indicates the simplifications which the use of Ricci's methods of covariant differentiation permit to be made in Forsyth's recent work on differential invariants of quadratic differential forms.

23. The main part of Professor Huntington's paper gives a systematic elementary account of the purely ordinal theory of the continuum, as begun by Dedekind in 1872, and completed by Georg Cantor in 1895. An appendix on Cantor's "well-

ordered" classes and the transfinite numbers will serve as an introduction to the study of these most recent accessions to the list of mathematical concepts. The matter is of interest to the philosopher as well as to the mathematician; and the present exposition is intended especially for the general student of scientific method, who, without technical mathematical training, wishes to keep in touch with the modern development in the logic of mathematics. The mathematical prerequisites have been reduced to a minimum; the demonstrations are given in full; all new concepts are defined explicitly by sets of independent postulates; and in connection with each definition numerous examples are given, to illustrate, in a concrete way, not only the systems which have, but also those which have not, the property in question. The paper is being published in the *Annals of Mathematics* for July and October, 1905.*

The chapter headings are as follows: On classes in general; Ordered classes, or "series"; Discrete series, especially the type of order exhibited by the natural numbers; Dense series, especially the type of the rational numbers; Continuous series, especially the type of the real numbers; Continuous series in more than one dimension, with a note on multiply-ordered classes. An appendix treats of Cantor's "well-ordered" classes, and the transfinite numbers, and there is an index of technical terms. The paper contains also a bibliography of Cantor's writings on these subjects.

F. N. COLE,
Secretary.

A SET OF GENERATORS FOR TERNARY LINEAR GROUPS.

BY MISS IDA MAY SCHOTTENFELS.

(Read before the American Mathematical Society, September 17, 1904.)

The following is a proof that the substitutions

$$c_h: \quad x_i' = x_{i+1},\ x_k' = x_1 + hx_2 \quad (i = 1, 2, \cdots k-1;\ h = 0, 1)$$

generate (1) the ternary linear substitution group with integral

* Reprints of this and other papers published in the *Annals* can be ordered from the Publication Office of Harvard University.

coefficients of determinant unity, and (2) the linear and linear-fractional Galois field groups $[2^1]$.*

In recent literature on linear and linear-fractional congruence groups, entrance to such groups has frequently been made by means of the above substitutions.†

In the proof of (1) the methods of Burnside ‡ are employed, and in (2) an analogue of this method is made use of for the Galois field $[2^1]$.

1. All substitutions of the form

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3,$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3,$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3,$$

with coefficients rational integers of determinant unity can be formed by combinations of the two following substitutions:

$$T = (x_2 \cdot x_3 \cdot x_1), \qquad S = (x_2 \cdot x_3 \cdot x_2 + x_1).$$

The following combinations only are made use of in this proof:

$$P = TST = (x_1 \cdot x_2 + x_3 \cdot x_3), \quad Q = T^2S = (x_1 \cdot x_2 \cdot x_3 + x_1),$$
$$U = PV^{-1}P = (x_1 \cdot x_3 \cdot - x_2), \; W = ST^{-1} = (x_1 + x_2 \cdot x_2 \cdot x_3),$$
$$V = (T^2S^2T^{-1})^2(ST^{-1})^{-2}(T^2S)^{-2} = (x_1 \cdot x_2 \cdot x_3 + x_2).$$

From the form of P, Q, W, V and T^nVT^{2n} $(n = 0, 1, 2)$ it is evident that T and S give rise to six substitutions in which two of the symbols are unchanged and the third is reinforced by either of the unchanged symbols.

From the form $T^nU^mT^{2n}$ $(n = 0, 1, 2; m = 1, 2, 3)$, derived from U, it is evident that T and S give rise to all substitutions in which one of the symbols remains unchanged.

The remainder of this proof consists in showing that corresponding to any substitution whatever, say Σ, a substitution

* Krazer, *Annali di Matematica*, (2) vol. 12, pp. 283–300. Kronecker, *Monatsberichte*, Berlin Akademie, Oct. 15, 1866, p. 597.

† Moore, "Concerning the general equations of the seventh and eighth degrees." *Mathematische Annalen*, vol. 51, p. 436. Schottenfels, *Annals of Mathematics*, 2d ser., vol. 1, No. 3; BULLETIN, 2d ser., vol. 6, pp. 440–443.

‡ Burnside, *Messenger of Mathematics*, vol. 24, p. 109.

can be formed by a finite number of combinations and repetitions of T and S which, performed after Σ, leads to identity, and which is therefore Σ^{-1}. The inverse of Σ^{-1}, which in turn can be formed by a finite number of combinations and repetitions of T and S, is therefore equal to Σ.

Let

$$\Sigma = (\alpha x_1 + \beta x_2 + \gamma x_3,\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3)$$

where $\gamma \geqq \gamma'$.

$$\Sigma W^m = ([\alpha + m\alpha']x_1 + [\beta + m\beta']x_2 + [\gamma + m\gamma']x_3,$$
$$\alpha' x_1 + \beta' x_2 + \gamma' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

or

$$\Sigma W^m = (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,$$
$$\alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

where the integer m, positive or negative, can always be chosen so that γ_1 is numerically less than γ'. Hence we have

$$\Sigma W^m (T^2 VT)^{m'} = (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,\ \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3,$$
$$\alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

where m' can be chosen so that γ_2 is numerically less than γ_1.

After a finite number of such substitutions, a substitution must be arrived at in which one of the two coefficients corresponding to γ and γ' in Σ is zero; and, if it is the coefficient corresponding to γ', combining with T^2UT gives a substitution in which the coefficient corresponding to γ vanishes and Σ is reduced to

$$\Sigma' = (ax_1 + bx_2,\ a' x_1 + b' x_2 + c' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3).$$

A precisely similar series of substitutions will reduce this to a form in which the coefficient corresponding to c' vanishes; and on the other hand since the determinant of the substitution remains unity, the coefficient corresponding to γ'' must become ± 1. By combining with U^2 if necessary, this last coefficient may be made $+1$, and since the coefficients a and b are unaffected by this set of substitutions, Σ' becomes

$$\Sigma'' = (ax_1 + bx_2,\ a_1 x_1 + b_1 x_2,\ a_2 x_1 + b_2 x_2 + x_3).$$

By repeating a set of substitutions similar to the first set, Σ'' may be reduced to a substitution in which b vanishes, and b_1 becomes $+1$; and since the determinant remains unity, a at the same time becomes $+1$, and Σ'' becomes

$$\Sigma''' = (x_1,\ a_1'x_1 + x_2,\ a_2x_1 + b_2x_2 + x_3).$$

Finally $(T^2UT)^{-a_1'}Q^{-a_2}.\ V^{-b_2}$ reduces Σ''' to

$$\Sigma^{IV} = (x_1,\ x_2,\ x_3) \text{ or identity.}$$

2. All matrices of the form $(\alpha_{ij})\,(ij = 1, 2, 3)$

$$\begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{matrix}$$

where α_{ij} are marks of the Galois Field $[2^1]$ and where the matrix (α_{ij}) is such that its determinant is unity can be formed by combinations of the two following matrices

$$T = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad S = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

The following combinations only are made use of in this proof:

$$P = TST = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \qquad Q = T^2S = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \qquad W = ST^{-1} = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$V = (T^2S^2T^{-1})^2 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad U = PV^{-1}P = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

As in case (1), P, Q, W, V and the form $T^nVT^{2n} (n = 0, 1, 2)$ give rise to six substitutions in which two of the symbols are unchanged and the third is reinforced by either of the unchanged symbols. Also the form $T^nUT^{2n} (n = 0, 1, 2)$ contains all the substitutions in which one of the symbols remains unchanged.

The remainder of this proof proceeds as in case (1). Let

$$\Sigma = \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix};$$

if $\gamma < \gamma'$, then $\gamma = 0$ and Σ becomes

$$\Sigma' = \begin{matrix} \alpha & \beta & 0 \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix};$$

otherwise $\gamma \geqq \gamma'$.

$$\Sigma W = \begin{matrix} \alpha+\alpha' & \beta+\beta' & \gamma+\gamma' \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix} \quad \text{or} \quad \Sigma W = \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}.$$

(1) If $\gamma = \gamma'$, then $\gamma_1 = 0$. (2) If $\gamma > \gamma'$, then $\gamma' = 0$.

In (1) ΣW reduces to Σ'. In (2) we combine ΣW with T^2UT and again get

$$\Sigma' = \Sigma W T^2 U T = \begin{matrix} \alpha' & \beta' & \gamma' \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha'' & \beta'' & \gamma'' \end{matrix},$$

where $\gamma' = 0$,

$$\Sigma' P = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha_1+\alpha'' & \beta_1+\beta'' & \gamma_1+\gamma'' \\ \alpha'' & \beta'' & \gamma'' \end{matrix} \quad \text{or} \quad \Sigma' P = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha'' & \beta'' & \gamma'' \end{matrix}.$$

If $\gamma_1 < \gamma''$, then $\gamma_1 = 0$, and Σ' becomes

$$\Sigma'' = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha_1 & \beta_1 & 0. \\ \alpha'' & \beta'' & \gamma'' \end{matrix}$$

In the remaining case $\gamma_1 \geqq \gamma''$.
(1) If $\gamma_1 = \gamma''$ then $\gamma_2 = 0$. (2) If $\gamma_1 > \gamma''$ then $\gamma'' = 0$.

In (1) $\Sigma' P$ reduces to Σ''. In (2) $\Sigma' P$ combined with U gives

$$\Sigma'' = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

$$\Sigma'' W = \begin{matrix} \alpha' + \alpha'' & \beta' + \beta'' & 0 \\ \alpha'' & \beta'' & 0, \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \quad \text{or} \quad \Sigma'' W = \begin{matrix} \alpha_3 & \beta_3 & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

If $\beta' < \beta''$ then $\beta' = 0$ and Σ'' becomes

$$\Sigma''' = \begin{matrix} \alpha' & 0 & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

In the remaining case $\beta' \geqq \beta''$.

(1) If $\beta' = \beta''$ then $\beta_3 = 0$. (2) If $\beta' > \beta''$ then $\beta'' = 0$.

In (1) $\Sigma'' W$ reduces to Σ'''. In (2) $\Sigma'' W$ combined with T^2UT gives

$$\Sigma''' = \begin{matrix} \alpha'' & 0 & 0 \\ \alpha_3 & \beta_3 & 0, \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

Finally $\Sigma''' T^2 UTQV$ gives Σ^{IV}, which is identity,

$$\Sigma^{IV} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0. \\ 0 & 0 & 1 \end{matrix}$$

In conclusion, the generators T and S give rise to the simple group of order 168 with generational definition as follows:

$$G_{168} = \{T^3 = S^7 = (ST^2)^2 = (TS^2)^4 = I\}.*$$

NEW YORK, *June*, 1905.

*Schottenfels: "Upon the non-isomorphism of two simple groups of order 8!/2 or 20160," BULLETIN, 2d ser. vol. 8, no. 1, p. 26, § 2.

NOTE ON THE STRUCTURE OF HYPER-COMPLEX NUMBER SYSTEMS.

BY DR. SAUL EPSTEEN.

(Read before the American Mathematical Society. September 7, 1905.)

IN the *Transactions* for April, 1905, on page 176 * the following theorem (No. III.) was enunciated:

Let $E, E_1, E_2, \cdots$ be a normal series of subalgebras of E (i. e., E_r is a maximal invariant subalgebra of E_{r-1}, $E_0 = E$) and let $K_1, K_2, \cdots$ be a series of complementary algebras, such that K_r accompanies E_{r-1} and is complementary to E_r. Under these assumptions the series $K_1 K_2 \cdots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \cdots$. In other words, if $E, E_1', E_2', \cdots$ is any other normal series, the complementary series of algebras $K_1', K_2', \cdots$ which it defines is the same as the series $K_1, K_2, \cdots$, apart from the sequence.

With the exception of this theorem and the corresponding one where a *chief series* is used in place of the *normal series* all the proofs were made in a symbolic notation which is a generalization of one introduced by Frobenius in the theory of abstract groups. † In demonstrating the above proposition we made use of the *characteristic constants* $\gamma_{i_1 i_2 i_3}$ of the system

$$E \equiv e_1 \cdots e_n \qquad (e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3}),$$

which, owing to the associative law, satisfy the n^3 conditions

$$\sum_{i_3} (\gamma_{i_1 i_2 i_3} \gamma_{i_3 i_4 i_5} - \gamma_{i_1 i_4 i_5} \gamma_{i_2 i_4 i_3}) = 0 \quad (i = 1, \cdots, n).$$

In the present note it is shown how this theorem can also be demonstrated in the symbolic notation without recourse to the γ's.

In order to avoid repetitions I employ the same numbering of theorems and equations, and also the same letters and symbols as in the above quoted paper, which this note supplements.

According to the demonstration of the theorem I, we have (if $E_1 \neq E_1'$)

$$E = E_1 + E_1'.$$

* Epsteen-Wedderburn, "On the structure of hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6, pp. 172–178.

† Frobenius, *Berliner Sitzungsberichte*, 1895, p. 164.

Let

$$E_1 \frown E_1'' = F_1;$$

then

$$E_1 = F_1 + D_1 \qquad (F_1 \frown D_1 = 0),$$

$$E_1'' = F_1 + D_1' \qquad (F_1 \frown D_1' = 0),$$

and it is evident that

$$E = F_1 + D_1 + D_1'.$$

By theorem 2, F_1 is a maximal invariant subalgebra of E_1 and E_1''.

If $F_1, F_2, F_3, \cdots$ is a normal series of F_1, then $E, E_1, F_1, F_2, \cdots$ and $E, E_1'', F_1, F_2, \cdots$ are two new normal series of E. For this purpose it is merely necessary to prove that the complementary algebras defined by E, E_1, F_1 and E, E_1'', F_1 are the same.

The complementary algebra of $E_1 (= F_1 + D_1)$ with respect to E is

$$(9') \qquad \begin{aligned} E(\text{mod } E_1) &= F_1 + D_1 + D_1' (\text{mod } F_1 + D_1) \\ &= D' (\text{mod } F_1 + D_1) = D_1' (\text{mod } F_1), \end{aligned}$$

since

$$(F_1 + D_1)^2 \equiv F_1 + D_1 \text{ and } D_1'^2 \frown D_1 = 0.$$

Furthermore, the complementary algebra of F_1 with respect to E_1 is

$$(10') \qquad E_1 (\text{mod } F_1) = F_1 + D_1 (\text{mod } F_1) = D_1 (\text{mod } F_1).$$

Now it is seen in the same manner that the complementary algebra of E_1'' with respect to E is

$$(10') \qquad D_1 (\text{mod } F_1)$$

and that the complementary algebra of F_1 with respect to E_1'' is

$$(9') \qquad D_1' (\text{mod } F_1)$$

and, therefore, the normal complementary series defined by E, E_1, F_1 is identical, apart from the sequence, with the normal complementary series defined by E, E_1'', F_1.

It now becomes necessary to show that the normal series E, E_1, $E_2, \cdots$; E, E_1, F_1, $F_2 \cdots$ define the same complementary series, and similarly for E, E_1', $E_2', \cdots$; E, E_1', F_1, $F_2, \cdots$. This amounts to proving the theorem for the algebras E_1 and E_1', which involves a finite number of repetitions of the above proof.

A chief or principal series E, P_1, $P_2 \cdots$ of an algebra being defined as one in which P_s is the maximal subalgebra of P_{s-1} which is invariant is $E(P_0 = E)$ it is easily shown in the symbolic notation, by exactly the same process as for the normal series, *that the system of complementary algebras* C_1, $C_2, \cdots$ *is independent of the chief series selected.*

In the case of the normal series the complementary algebras K_1, $K_2 \cdots$ are necessarily simple, but this is not true of the complementary algebras C_1, $C_2 \cdots$ in the case of the chief series.

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A GEOMETRIC PROPERTY OF THE TRAJECTORIES OF DYNAMICS.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, September 7, 1905.)

SUPPOSE that the force acting on a particle whose coordinates are x, y, z, produces an acceleration having the components $\phi(x, y, z)$, $\psi(x, y, z)$, $\chi(x, y, z)$. The equations of motion are then

$$(1) \qquad \ddot{x} = \phi(x, y, z), \ \ddot{y} = \psi(x, y, z), \ \ddot{z} = \chi(x, y, z),$$

where dots denote differentiation with respect to the time t. In such a field of force the initial position and initial velocity completely determine the trajectory. The totality of trajectories thus constitutes a quintuply* infinite system of space curves.

Consider now those trajectories obtained by starting the par-

* The only exception arises in the trivial case where the force is everywhere zero. Then the trajectories are the fourfold infinity of straight lines.

ticle at a given point P in a given direction. These form a simply infinite system since the initial speed v is arbitrary.

The osculating plane at P must contain both the given direction and the direction of the force acting at P; hence the ∞^1 trajectories have the same osculating plane. The osculating sphere, however, varies from trajectory to trajectory. We now prove that, no matter what the law of force, the locus of centers of these ∞^1 osculating spheres is a straight line.

The formulas for the center of the osculating sphere of any space curve are†

$$\begin{aligned} X &= x + r^2x'' - \rho r r'(y'z'' - z'y''), \\ Y &= y + r^2y'' - \rho r r'(z'x'' - x'z''), \\ Z &= z + r^2z'' - \rho r r'(x'y'' - y'x''), \end{aligned} \tag{2}$$

where accents denote differentiation with respect to the arc s; r is the radius of curvature, and ρ the radius of torsion. These are determined by the familiar formulas

$$1/r^2 = x''^2 + y''^2 + z''^2 = \Sigma x''^2, \tag{3}$$

$$\frac{1}{\rho} = -\frac{1}{\Sigma x''^2}\begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix}. \tag{4}$$

Differentiating (3), we find

$$r' = -r^3\Sigma x''x'''. \tag{5}$$

To apply these formulas to the trajectories of dynamics, it is necessary to replace arc derivatives by time derivatives. The transformation formulas are

$$\begin{aligned} x' &= \dot{x}/v, \\ x'' &= (v\ddot{x} - \dot{v}\dot{x})/v^3, \\ x''' &= [v(v\dddot{x} - \ddot{v}\dot{x}) - 3\dot{v}(v\ddot{x} - \dot{v}\dot{x})]/v^5, \end{aligned} \tag{6}$$

with similar results for y and z.

† See Scheffers, Theorie der Curven, Leipzig (1901), p. 234.

Here v denotes the velocity, so that

$$(7) \qquad v^2 = \Sigma \dot{x}^2, \quad v\dot{v} = \Sigma \dot{x}\ddot{x}, \quad v\ddot{v} + \dot{v}^2 = \Sigma \dot{x}\dddot{x} + \Sigma \ddot{x}^2.$$

By means of formulas (2) to (7), the coördinates of the center of the osculating sphere may be expressed in terms of x, y, z and their time derivatives of first, second and third orders.

We now simplify the discussion by means of a particular choice of axes. Let the given point P be taken as origin; the given direction, as axis of x; and let the fixed osculating plane be the plane xy. Since the initial velocity is directed along the axis of x, we have

$$\dot{x},\ \dot{y},\ \dot{z},\ = v,\ 0,\ 0.$$

The force acts in the plane xy, hence the acceleration components are

$$\ddot{x},\ \ddot{y},\ \ddot{z} = \phi,\ \psi,\ 0,$$

where ϕ, ψ are constants since $x,\ y,\ z,\ = 0,\ 0,\ 0$. The third derivatives are

$$\dddot{x},\ \dddot{y},\ \dddot{z},\ = v\phi_x,\ v\psi_x,\ v\chi_x.$$

From (6), we can now find the arc derivatives

$$(8) \qquad \begin{aligned} &x',\ y',\ z' = 1,\ 0,\ 0,\\ &x'',\ y'',\ z'' = 0,\quad \psi/v^2,\ 0,\\ &x''',\ y''',\ z''' = (-\psi^2,\ v^2\psi_x - 3\phi\psi,\ v^2\chi_x)/v^4. \end{aligned}$$

Substituting these values in (3), (4), (5), we have

$$(9) \qquad r = \frac{v^2}{\psi}, \quad \rho = -\frac{\psi}{\chi_x}, \quad r' = \frac{3\phi\psi - v^2\psi_x}{\psi^2}.$$

Finally, from (2), we find

$$(10) \qquad X = 0,\ Y = \frac{v^2}{x},\ Z = \frac{3\phi\psi - v^2\psi_x}{\psi\chi}$$

Here we note that X, Y, Z are linear functions of the parameter v^2. Elimination thus gives the straight line

$$(11) \qquad X = 0\,,\ Z = \frac{3\phi}{\chi_x} - \frac{\psi_x}{\chi_x}\,Y$$

as the required locus of centers.

Incidentally, the value of ρ obtained in (9) shows that the torsion, unlike the curvature, is independent of v.

An arbitrary field of force (1) *produces* ∞^5 *trajectories, of which* ∞^1 *pass through a given point in a given direction. These* ∞^1 *trajectories have, at the given point, a common osculating plane and a common torsion. The locus of centers of their osculating spheres is a straight line. Thus every field of force gives rise to a correspondence between the direction elements and the straight lines of space.*

COLUMBIA UNIVERSITY,
August, 1905.

ON THE POSSIBLE NUMBERS OF OPERATORS OF ORDER 2 IN A GROUP OF ORDER 2^m.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 7, 1905.)

It is well known that every group of order 2^m which contains only one operator of order 2 is either cyclic or it is composed of the cyclic group of order 2^{m-1} and 2^{m-1} operators of order 4 transforming each operator of this cyclic group into its inverse.* There are exactly two such groups for every value of $m > 2$. When $m = 3$ the latter of these two is the quaternion group, and when $m < 3$ the cyclic group is the only one that contains only one operator of order 2.

The groups of order 2^m in which the number of all the operators of order 2 is $\equiv 1 \bmod 4$ have been determined incidentally in a recent paper.† Such groups exist only when the number of operators of order 2 is of the form $2^k + 1$, and there are exactly two possible groups for every arbitrary value of k. One of these is the dihedral rotation group of order 2^{k+1}, and the other is obtained by adding to the cyclic group of order 2^{k+1} an operator of order two which transforms each of its operators into its $(2^k - 1)$th power. Just half of the additional operators are of order two and the others are of order 4.

For instance, there are just two groups whose orders are of the form 2^m and which contain just five operators of order two;

* Burnside, Theory of groups, 1897, p. 75.
† *Transactions Amer. Math. Society*, vol. 6 (1905), p. 62.

one of these is the well known octic group, or the group of movements of the square, while the other is of order 16 and includes the cyclic group of order 8 in addition to four operators of each of the orders 2 and 4 transforming each operator of this cyclic subgroup into its third power. The latter constitute two complete sets of conjugates, while the operators whose orders exceed two in the cyclic subgroup of order 8 are conjugate in pairs. In general, the operators whose orders exceed two in the cyclic subgroup of orders 2^k or 2^{k+1} respectively are conjugate in pairs while the remaining operators constitute two complete sets of conjugates.

It is well known that every group whose order is divisible by 2 contains an odd number of operators of order 2. Moreover, it is clearly possible to construct a group which contains any arbitrary odd number of operators of order 2. To construct a group which contains exactly $2n + 1$ operators of this order it is only necessary to extend the cyclic group of order $2n + 1$ by an operator of order 2 which transforms each operator of this cyclic group into its inverse. From the theorem mentioned in the second paragraph it follows that it is impossible to construct a group whose order is of the form 2^m and which contains an arbitrary odd number of operators of order 2.

In view of the popular saying that 13 is an unlucky number, it is of interest to note that *there is no group whose order is of the form* 2^m *and which contains exactly* 13 *operators of order* 2, but there are groups which contain exactly any other odd number, less than twenty-one, of such operators. The truth of this statement follows almost directly from the second paragraph. It will be included among the results of the developments which follow. In these developments it will be assumed that the orders of all the groups under consideration are powers of 2.

The dihedral rotation group of order 2^{k+1} contains exactly $2^k + 1$ operators of order 2, whenever $k > 0$. By forming the direct product of this dihedral rotation group and the abelian group of order 2^β and of type $(1, 1, 1, \cdots)$ there results a group in which the number of operators of order 2 is exactly $3^{k+\beta} + 2^{\beta+1} - 1$. As the values of β and k are arbitrary except that $k \neq 0$ it follows that *it is possible to construct groups in which the number of operators of order* 2 *is exactly* $2^x + 2^y - 1$, *where* x *and* y *can have any arbitrary pair of integral values with the exception of* 0, 1. When the groups are abelian $x = y$, but the converse is not necessarily true.

From the preceding paragraph it follows that it is possible to construct a group of order 2^m in which the number of operators of order 2 is the sum of two arbitrary powers of 2 diminished by one. We shall now prove that it is possible to construct a group in which the number of these operators is the sum of three arbitrary powers of two diminished by one, provided the smallest of these powers is divisible by 4. It is clear that we may restrict ourselves to the case when the three powers are distinct since the other cases reduce to those considered in the preceding paragraph.

To prove this theorem it is convenient to consider the group (G_1) of order $2^{\alpha+2}$ which is obtained by taking all the operators of the holomorph of the cyclic group of order 2^α which transform this cyclic group into its $1, 2^{\alpha-1}-1, 2^{\alpha-1}+1$, and $2^\alpha-1$ powers respectively. These powers are distinct when $\alpha > 2$. In this case there are $1, 2^{\alpha-1}, 2$ and 2^α operators of order 2 which transform the cyclic group into the respective powers mentioned above. It is clearly possible to establish a $(2^\alpha, 2^\beta)$ isomorphism between G_1 and the dihedral rotation group of order $2^{\beta+2}$ in such a manner as to obtain a group (G) which contains $3 + 2^{\beta+1} + 2^{\alpha+\beta}$ operators of order 2, whenever $\beta > 0$.

As α and β are arbitrary independent positive integers, with the restriction that $\alpha > 2$, it follows that the number of operators of order 2 in G is the sum of two powers of 2, whose indices differ by at least two units but are otherwise arbitrary, increased by 3. In G_1 the number of these operators is the sum of two powers of 2, whose indices differ by one but are otherwise arbitrary, increased by 3. Hence by forming the direct product of an abelian group of order 2^γ and of type $(1, 1, 1, \cdots)$ into either G or G_1 we obtain *a group in which the number of operators of order* 2 *is* $2^x + 2^y + 2^z - 1$, *where* x, y, z *satisfy the condition* $x > y > z > 1$ *are but otherwise arbitrary.* This proves the theorem in question.

It has recently been proved that the number of operators of order 2 in a group of order 2^m, in which more than half the operators are of order 2, is always the sum of two powers of 2 diminished by one, and that it is possible to construct such a group in which the number of operators of order 2 is an arbitrary number of this form. Hence it follows from the preceding paragraph that groups in which less than half the operators are of order 2 may involve a number of operators of this order which could not be the exact number of these opera-

tors in any group in which more than half the operators are of order 2.

A $(2^\alpha, 2^\beta)$ isomorphism between G_1 and the direct product of the dihedral rotation group of order $2^{\beta+1}$ into an operator of order 2 can be established in such a manner as to obtain a group in which the number of operators of order 2 is either $3 + 2^\alpha + 2^{\beta+1} + 2^{\alpha+\beta}$, or $3 + 2^{\alpha+1} + 2^{\beta+1} + 2^{\alpha+\beta-1}, \beta > 0$. In fact, it is possible to form other such isomorphisms, but these two seem especially useful in this connection. Moreover, by establishing a $(2^\alpha, 2^\beta)$ isomorphism between G_1 and a group of order $2^{\beta+2}$ which is constructed in the same way as G_1, we arrive at groups which contain any of the following three numbers of operators of order 2: $3 + 2^\alpha + 2^\beta + 2^{\alpha+\beta}$, $3 + 2^{\alpha+1} + 2^{\beta+1} + 2^{\alpha+\beta-2}$, $3 + 2^{\alpha+1} + 2^\beta + 2^{\alpha+\beta-1}$.

From the above results it follows directly that there are groups of order 2^m which contain any prescribed number of operators of order 2 which satisfies the conditions that it is $\equiv 3$ mod. 4 and less than 124. By other considerations this limit can readily be extended, but my methods seem too special to be given here. It would be interesting to find a number $\equiv 3$ mod. 4 which could not equal the number of operators of order 2 in any group of order 2^m, or to prove the non-existence of such a number.

ON THE ARITHMETIC NATURE OF THE COEFFICIENTS IN GROUPS OF FINITE MONOMIAL LINEAR SUBSTITUTIONS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 7, 1905.)

PROFESSOR MASCHKE* has proved (with a certain restriction) that the coefficients of finite linear substitution groups can, by proper transformations, be made rational functions of roots of unity. Professor Burnside† has also recently written on this subject. In this note it is proved that linear groups all of

* Maschke, *Math. Annalen* v. 50 (1898), p. 492.
† Burnside, *Proc. London Math. Society*, ser. 2, v. 3 (1905), p. 239.

whose elements are finite monomial substitutions, that is, of the form

$$x_i' = a_j x_j \qquad (i, j = 1, 2, \cdots, n),$$

(n is the number of variables) can be written so that all the coefficients are *roots of unity.*

THEOREM I. *Any non-vanishing element of the principal diagonal of a monomial substitution of finite period is a root of unity.*

Consider the kth power of a substitution S. If a lies in the principal diagonal of S, a^k occupies the same place in the principal diagonal of S^k. Hence a is a root of unity.

THEOREM II. *The product of two non-vanishing elements of the same or any two substitutions of a group (G) of finite monomial linear substitutions, which are symmetrically placed with respect to the principal diagonal, is a root of unity.*

The product of these two elements stands in the principal diagonal of the product of the two substitutions.

THEOREM III. *If G has no coefficient a_{ik} zero for all its substitutions, it can be transformed into another monomial group such that the non-vanishing element in the first column of every substitution of G becomes a root of unity.*

Since no element a_{ik} is zero for every substitution of G, we can choose $n - 1$ substitutions

$$A^{(2)} = (a_{ik}^{(2)}), \quad A^{(3)} = (a_{ik}^{(3)}), \cdots, A^{(n)} = (a_{ik}^{(n)}),$$

in which none of the coefficients

$$a_{12}^{(2)}, \ a_{13}^{(3)}, \ \cdots, \ a_{1n}^{(n)}$$

vanish. Transform G by the canonical substitution

$$x_i' = x_i / p_i \qquad (i = 1, 2, \cdots, n).$$

The transformed group (G') is monomial. If p_1 is an arbitrary root of unity, and if $p_2 = a_{12}^{(2)}, p_3 = a_{13}^{(3)}, \cdots, p_n = a_{1n}^{(n)}$, we have in place of $A^{(2)}, A^{(3)}, \cdots, A^{(n)}$ substitutions with p_1 as the only non-vanishing element in the first row of each. Now apply Theorem II to all the substitutions of G', and the present theorem follows.

THEOREM IV. *If G' has no coefficient everywhere zero the non-vanishing elements of every substitution are roots of unity.*

Consider the first column of a product AB. The elements of this column are obtained by multiplying the rows of A into the first column of B. Let A and B run through all the substitutions of G. Every coefficient is seen to be the quotient of two roots of unity, that is, a root of unity.

Suppose that a certain coefficient a_{ik} vanishes in every substitution of G. We may assume that the variables of G have been so permuted that the $n-r$ last elements in the first row of all the substitutions of G are zero, and that no other row has more than $n-r$ elements which vanish for every substitution of G.

THEOREM V. *Every substitution of G is in the form*

$$\left[\begin{array}{c|c} N_1 & \begin{matrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdot \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdot \end{matrix} & N_2 \end{array}\right],$$

where N_1 and N_2 are monomial matrices, without further transformation.

There now are $r-1$ substitutions $A^{(2)}$, $A^{(3)}$, $\cdots$, $A^{(r)}$ in which the coefficients $a_{12}^{(2)}$, $a_{13}^{(3)}$, $\cdots$, $a_{1r}^{(r)}$ do not vanish. From the products

$$A^{(2)}B,\ A^{(3)}B,\ \cdots,\ A^{(r)}B,$$

$$a_{12}^{(2)}b_{2i}=0,\ a_{13}^{(3)}b_{3i}=0,\ \cdots,\ a_{1r}^{(r)}b_{ri}=0 \quad (i=r+1,\cdots,n),$$

where B is any substitution of G. Hence the last $n-r$ coefficients of the first r rows of all the substitutions of G are zero. Since these substitutions are monomial the first r elements in the last $n-r$ rows are also everywhere zero.

The group in the variables $x_1, x_2, \cdots x_r$ has by hypothesis no coefficients that are everywhere zero, so that for it Theorem IV holds.

We continue in this way with the group in the last $n-r$ variables, and finally have the

THEOREM VI. *The coefficients of a group of monomial linear substitutions of finite period may, by means of transformations which leave them monomial, be made roots of unity.*

STANFORD UNIVERSITY,
August 7, 1905.

A MODERN CALCULUS OF VARIATIONS.

Lectures on the Calculus of Variations. By OSKAR BOLZA. (The University of Chicago Decennial Publications, Second Series, Volume XIV.) The University of Chicago Press, Chicago, 1904. 8vo, xv + 271 pp.

THE calculus of variations is one of the very first of the old (or formal) developments of the infinitesimal calculus; it is one of the newest conquests of the modern (or critical) school. The history of the older calculus of variations is almost trite from reiteration; to select among many, the works of Moigno-Lindelöf, Diegner, Todhunter, Carll, Jordan, and (more recently and more perfectly) Pascal, have made known the achievements of the old school from Newton to Jacobi to mathematicians of all nations. The problems and the successes of the new school — the critical investigations — have been known to few until this century opened. In 1900 the first really modern treatise — Kneser's Lehrbuch — appeared. Without gainsaying the importance of that work and the fact that it opened the doors of modern research in the calculus of variations to the general mathematical public, it must be admitted that its style and arrangement are forbidding. The articles in the Encyklopädie (by Kneser and by Zermelo and Hahn) scarcely dispel the gloom of the Lehrbuch's unilluminated interior; but this was not to have been expected in encyclopedia articles, and the circumstances which necessitated two separate articles did not particularly conduce to public enlightenment on comparative values of various methods.

The book which is the subject of the present review is of interest to us because it is the first accurate and critical presentation of all the modern methods which is thoroughly readable, and because of the natural favorable prejudice of language and country. Apart from any such bias, however, Professor Bolza's treatise is surely admirable in many ways. Let us examine some of its notable features.

The preface is illuminating concerning much that has not been generally known, and the statements of the essential differences between the older theory and the new are worthy of recapitulation. They are: (1) the critical revision, by Weierstrass, Erdmann, Du Bois-Reymond, Scheefer, Schwarz, and others, of results previously obtained; (2) Weierstrass's intro-

duction of the parameter representation; (3) Weierstrass's discovery of the fourth necessary condition and the sufficient conditions, with many subsequent allied contributions by others; (4) Kneser's extension to extremals of the theorems on geodetics; (5) Hilbert's *a priori* existence proof. That this is given at some length here and that the body of the book is often tinged with a historical coloring is doubtless due to the fact that these Lectures are a development of the Colloquium Lectures delivered by Professor Bolza at the Ithaca meeting of the AMERICAN MATHEMATICAL SOCIETY (August, 1901), which will be remembered with pleasure by many readers.

In the opening pages (§ 1, Introduction; § 2, Notation and terminology) one is made conscious of the rigorous nature of the discussions to be expected, but is reassured by the conciseness and clearness of statement, which go to make the book throughout easily readable to one otherwise familiar with exact reasoning. The reviewer would call attention to the precise definitions of interval, region, continuum, etc., on page 5. Some general agreement concerning these terms was and is needed, and the definitions here given may serve as a basis, even if they are not generally adopted verbatim. A difference of opinion is more tenable here, however, than in the demonstrations of later pages. Thus it might be better to reserve the name region for what is called a closed region, since the term continuum already connotes an absence of boundary points, and the word domain already serves as a generic term including closed regions, continua, and the intermediate cases. An interval, as defined here, is apparently always "closed."

The crucial article on the formulation of the problem (§ 3, page 9) leaves nothing to be desired. It should be noticed that maximum (or minimum) as here defined is that which is called "improper," in contradistinction to the case of "proper" maximum (or minimum). The usage in this matter must be settled by time. Although the present definition does not materially alter the conditions discovered by others under the other definition, it seems possible that it will change the nature of the necessary conditions which remain to be discovered (see page 100), and it would necessitate the recognition of any curve whatever as a minimizing curve in the case of integrals which are independent of the path. The use of the phrase "proper minimum" (see page 30, line 29) as distinguished from "minimum" may be misleading.

The next articles contain the two older classical derivations of the condition that the first variation should vanish (§ 4), Du Bois-Reymond's lemma which reduces the preceding condition to the *first necessary condition* (§ 5), and the Du Bois-Reymond-Hilbert proof of that condition without the assumption of the existence of the second derivative (§ 6). The presentations contain nothing essentially unfamiliar, except that the last proof is here presented for the first time in a separate work, but they are especially clear and the footnotes contain valuable remarks and references. The use of Euler's name instead of Lagrange's to designate the fundamental differential equation is novel and is apparently justified (footnote 3, page 22).

The miscellaneous remarks on the integration of Euler's equation in § 7 include several important matters, for example the proof of the existence of extremals in any direction not parallel to the Y axis (page 28), the case in which Euler's equation reduces to an identity (page 29), and the inverse problem (page 31). The formal side of the latter problem is outlined here in concise and satisfactory form, but it is unfortunately not reconsidered later after the development of the sufficient conditions.

The chapter closes with discussions of Weierstrass's lemma (§ 8), including the introduction of the E function and the concept of transversality; discontinuous solutions (§ 9); and (§ 10) the conditions at a boundary.

Chapter II is for the most part an account of the treatment of the so-called *second variation* by the methods of the old school. While the general lines of this work are familiar from older works, the nice distinctions and exact statements of this presentation are characteristic of the modern point of view and are not to be found in older works. We shall mention here only the following points: (1) the exact statement of Legendre's condition (page 47); (2) the exact statement of the assumptions (§ 13, pages 54–55) made in the proof of Jacobi's theorem; (3) the exact statement of Jacobi's condition (page 67). On the whole this chapter contains less that is modern, except in point of exactness, than Chapter I, but here also the notes and references are valuable even to one familiar with the older theory.

The study of the sufficient conditions in Chapter III formally opens the modern theory, though this would be impossible without the accurate revisions of the older theory in the previous chapters.

With respect to the author's view (page 70) that the concept of "weak" extrema, for which he has just proved that Jacobi's and Legendre's conditions are sufficient, is of only temporary importance, there may be differences of opinion. The distinction between "weak" and "strong" minima is of precisely the same kind as the distinction between "strong" minima and "absolute" minima, or as the distinction between isolated minima and absolute minima on a plane curve. In case no "strong" minimizing curve exists, for example, it seems interesting at least to discover the "weak" ones. Such is actually the case in Newton's famous problem on the surface of rotation of least resistance. Again, a "strong" minimizing curve is surely also a "weak" minimizing curve; hence we need seek for "strong" minimizing curves only among the "weak" minimizing curves.

Passing to the case of a strong minimum, Bolza first proves Weierstrass's (the fourth) necessary condition by means of Weierstrass's lemma. Then follows in §§ 19–20 the exquisite modern theory due to Weierstrass and Hilbert, resulting in Weierstrass's theorem and the first set of *sufficient* conditions. The matter presented is already fairly familiar to the American mathematical public through articles by Bolza himself, Osgood, and others, to which the most detailed references are given. But this is the first systematic treatment of the theory from all these points of view, and it is probably easier to compare the different methods here than in the isolated memoirs. It goes without saying that Kneser's treatment is much more one-sided here and in many other places, if for nothing else on account of the achievements of Hilbert and others, after the appearance of the Lehrbuch, which were considered to warrant the extra article in the Encyklopädie by Hahn and Zermelo. It seems to be conceded that the simplest and most elegant proof of Weierstrass's theorem is that based upon Hilbert's invariant integral, and subsequent treatments of the calculus of variations can probably afford to limit their discussion to a single proof. It might be advantageous to make even more use of Hilbert's principle than is indicated here. In particular the matter on page 266 (addenda) and that in footnote to page 175 indicate the possible important applications of this and similar methods.

The notation employed for the various conditions (I, II, IIa, IIa', etc.) is somewhat novel but ceases to be con-

fusing after careful examination, and the principal results are carefully stated in prominently lettered theorems. Thus the "Fundamental Theorem" (page 96), with the various appended notes (page 96; 1, page 99; and 1, p. 101), is sufficiently clear to dispel any confusion which might otherwise arise from such alternate statements as those on pages 98–101. The spirit of c), (page 98), shows the predilection of the author for the parameter representation of Weierstrass. Of course the omission of the equality signs from the necessary conditions to make them (apparently) coincide with the sufficient conditions is just as vital as any other difficult hiatus which might be suppressed as an "exceptional case," and a "fifth necessary condition" (see bottom of page 100) is just as indispensable in the case of parameter representation as it is here. But no such considerations are needed to establish the superiority of the parameter representation for a certain class of problems, nor could they possibly show a superiority for certain others — notably for certain problems of mechanics. The facts established in these pages (98–101) are nevertheless important and unquestionably rigorous, if we expressly repudiate any false inferences which a reader might draw upon his own account. Perhaps the theorem stated in footnote 1, page 101 is sufficiently important to have deserved a place in the text proper; but the reviewer has elsewhere* insisted upon this point sincerely enough and with possibly an excess of fervor.

Finally, the table of all the conditions on pages 101 and 102 removes any ambiguity which may remain. This table is for all practical purposes the end of the consideration of the simplest problem — namely that of minimizing the integral

$$\int_{x_0}^{x_1} f(x, y, y')dx$$

between given fixed end points.

Dividing the chapters as he does, Bolza has included in this chapter (III, Sufficient conditions) a discussion of one of the many possible generalizations — probably the first which naturally suggests itself — namely the case in which the end points are variable. The presentation is by the "method of the differential calculus," which reduces the new problem to a

* "Supplementary note on the calculus of variations," BULLETIN, vol. 9, No. 5 (February, 1905).

combination of the preceding one and that of ordinary maxima and minima, which may be assumed as known (see footnote, pages 102–103). This fact renders this discussion somewhat less interesting as an independent theory and augments the importance of the previous problem. The reviewer therefore feels that a rather disproportionate fraction of the volume has been given to this matter, in view of the fact that it is taken up again twice. The results are interesting, particularly in connection with Kneser's theory (Chapter V), of which it forms almost an integral part, but in view of the limited size of the book and the important omissions mentioned below the three separate treatments might well have been condensed into a single one, say under Chapter V.

Chapter IV is an exposition of the theory of the calculus of variations in parameter form, according to the methods of Weierstrass. The author is very frank, and apparently entirely just in his estimate (footnote, page 115) of the advantages of the method. The asymmetric forms of the preceding chapters apply in reality, as the author remarks, to those problems in which the solution $y = f(x)$ is thought of as a function rather than as a curve, but this implies that the asymmetric forms are to be preferred not only in the theory of functions of a real variable, but also in numerous applications which are not primarily geometric. One is often deceived by the convenient employment of geometric terms into a belief that the essence of a problem is geometric when it is not so, and the number of problems in which the parameter representation is superior might easily be overestimated.

On the other hand there is no possible question of the immense advantage of the parameter representation for the precise treatment of geometric problems. This advantage is shared with absolute equality, however, by the similar treatment of any geometric application which arises in any branch of mathematics whatever. This fact has been recognized but a short time even in the very elements of geometry — even in the definition of a curve. It is for those nice considerations which become possible in an accurate critical revision of a theory, and not for the relatively crude work of discovery of new and general principles, however, that these forms are valuable even in geometry, and there is many a geometric problem which has not yet been stated in this manner. The calculus of variations is no more a geometric subject than is many another branch of mathematics, and the stimulus which the success of these methods in the calculus of

variations will doubtless produce will surely cause the production of supplementary theories in these other subjects which will make applications to geometric problems as safe as are the applications of the results of this chapter. As a fitting example I might mention first of all the theory of differential equations, in which the ordinary theorems are no better suited to geometric applications than are those of the first three chapters of this book. A supplement to the theory of differential equations along the lines of the present chapter would be valuable and is indispensable to just the extent that the Weierstrass parameter representation is in the calculus of variations. As a matter of fact, on account of the perfect one-to-one correspondence of the real number system with the points upon a line and that of the assemblage of all pairs of real numbers with the points of a plane, we have been too unconscious of the discrepancy between curves on the one hand and equations of the form $y = f(x)$ on the other. It is perhaps not too much to say that geometric problems in general constitute by no means the nearest application of any well-known mathematical theory in the form in which it is at present usually given.

The development of the various conditions is wholly analogous to the work of previous chapters, and pages 115–146 and 153–156 involve few considerations which are *essentially* new; there are indeed new *facts*, but these are such as we should expect in any discussion in passing from non-homogeneous to homogeneous forms. In many cases omissions are made when the generalizations are rather obvious (see page 115, footnote 1). The following points deserve at least mention: (1) Weierstrass's form of Euler's equation (page 123) and the commentary upon it (page 124); (2) the proof (page 125) of the existence of an extremal in any direction; (3) the statement and proof of the necessary and the sufficient conditions (page 143); (4) the proof of the existence of a minimum "im Kleinen" (pages 146–147); (5) the conditions at a boundary (pages 148 ff.); and (6) the extension of the meaning of a definite integral (pages 156–163).

With regard to the statement of the sufficient conditions on page 143, the possibility of misunderstanding has already been mentioned. The reader should be careful to note that Bolza does not even imply that there is any one set of conditions given which is at once necessary *and* sufficient, though he seems to use those words. The four sufficient conditions stated on that page have nowhere been shown to be necessary, nor are they so (see footnote 2, page 148). While not wishing to criti-

cize the real content of these statements, the reviewer feels that the actual gap between the necessary and the sufficient conditions should not only not be lost sight of, it should be emphasized and brought home by every possible device; the conscious or unconscious failure to do this in the past has been the bane of exact mathematical reasoning. For this reason the reviewer would object to the notation which apparently identifies two different conditions, one of which is necessary and the other sufficient, and he would object to the phraseology (in *d*, page 143) which bears out this false impression. As a matter of fact the necessary and the sufficient conditions are not appreciably nearer together than they are in the other — non-geometric — problem; they are not appreciably nearer than they have been for several years.

The proof of the existence of a minimum "im Kleinen," which follows Bliss, *Transactions of the American Mathematical Society*, volume 5 (1904), is very important in many applications and constitutes one of the several important and wholly new topics treated. It is regretable that no mention is made of Hilbert's concept of "Stückweise Variation," * which is intimately connected with this topic and which has important applications.

The treatment of the extension of the meaning of a definite integral is also particularly noteworthy. The results of Weierstrass, Hilbert, Osgood, Peano and others are presented in their essentials and compared. The subject is again taken up in Chapter VII and forms an integral part of the author's excellent presentation of Hilbert's *a priori* existence proof.

The proof just mentioned is given in detail in Chapter VII, which we shall consider here on account of the connection mentioned above. The whole chapter is carefully written and contains the first accurate detailed proof of Hilbert's theorem. There is first a precise statement of the theorem and of the conditions under which it is to be proved. Then follows a supplementary investigation of extensions of the meaning of an integral, which together with the treatment just mentioned forms one of the important contributions of the book. I shall not enter into the details of the proof. It is rather long, but seems as concise as possible, and it actually proves, with apparently only necessary restrictions, all that was ever

* The reviewer has elsewhere translated this as "limited variation." See BULLETIN, vol. 9, p. 12.

claimed for the Hilbert proof in the case of a simple integral. In its accuracy and generality this chapter really amounts to an original contribution to mathematics, since neither Hilbert nor anyone else has published in any place an accurate proof of the theorem in its most general form, although portions of the proof and special cases of it have been proved by Hilbert and others and the truth of the theorem rather generally accepted. Leaving aside any other merit, this chapter alone would make this book a pioneer work of the first importance.

Chapters V and VI are devoted, respectively, to Kneser's transversal theory and the isoperimetric problem. Since the main features of both these chapters follow Kneser's Lehrbuch, they will be passed over rapidly.

Chapter V begins with the general explanation of Kneser's transversal theory as an extension of the ordinary theory of geodetic lines, and it is pointed out that this theory applies with peculiar force to the case in which the end points are variable. Kneser's principal theorems (on transversals, page 172; and on the envelope of a set of extremals, page 174) are given in § 33. It is shown in a note (page 175) that these theorems can be derived in a manner wholly analogous to Hilbert's original considerations which led to Hilbert's invariant integral theorem. These considerations are given in a note in the Addenda (page 266) in almost the same form in which Hilbert led up to his theorem in the course of lectures mentioned on pages 246 and 268. It should be mentioned that this course was formally entitled "Ausgewählte Capitel aus der Flächentheorie" and dealt with the calculus of variations *via* the problem of geodetic lines. The proof of the existence of a field (cf. Kneser, Lehrbuch, § 14) is made rigorous by the considerations of § 34, and the way is prepared for the introduction of Kneser's curvilinear coördinates. A number of important conclusions are stated in § 36, including Osgood's theorem "concerning a characteristic property of a strong minimum" (*Transactions of the American Mathematical Society*, volume 2, 1901). Kneser's proof of Weierstrass's theorem is compared with Hilbert's in § 37 and the unnecessary assumption in Kneser's proof is discussed in an enlightening manner. The chapter closes with a discussion of focal (or critical) points for the case of variable end-points.

The treatment of the isoperimetric problem follows Kneser's Chapter IV and Weierstrass's lectures. The essential facts —

Euler's rule, Meyer's law of reciprocity, the various necessary and sufficient conditions — are presented with the usual clearness and accuracy. In both these chapters which follow Kneser, the advantage over Kneser in clearness and conciseness is indisputable.

The reviewer is not one who would consciously conceal the faults, much less the errors, of a treatise under his criticism. But if there are serious errors in this book, they have escaped the reviewer either through ignorance or through insufficient scrutiny. The faults of a book are often matters of opinion, and I have pointed out a very few minor points in which I should venture to disagree with the notation or with the method of presentation. There are two other general criticisms of this nature. The arrangement of the matter into chapters was probably the result of careful consideration, but it seems that space might have been saved had the chapters been arranged according to subject matter rather than according to authors or the nature of the conditions to be proved. Thus Weierstrass's theorem, the case of variable end points, and several other matters are all considered in each of three distinct chapters. Again the repeated proofs of certain theorems by several methods consumes space, but this is made up for by the value of the comparisons which become possible. A future treatise will be spared this burden by reason of the existence of this book. As a result, perhaps, of the policy just mentioned, many important topics have been omitted altogether. For example, no attention is paid to the extension of the theory to double integrals, nor to the case in which higher derivatives enter in the integrand; and the consideration of the important inverse problem is restricted to the formal work on the two pages 31–32, which gives only the results due to Darboux.

These suggestions regarding arrangement and selection of topics do not alter the opinion already expressed that the treatment of the topics which are considered is excellent and that the book as a whole is an important contribution to mathematical knowledge not only in America but throughout the world. That Bolza's work is of international importance is shown by the speedy recognition it has found in at least one foreign quarter: the chapter on the calculus of variations in the second volume of Goursat's very recent Cours d'Analyse contains explicit acknowledgment of the indebtedness of its author to this treatise.

May this be a pioneer, not only in the subject which it represents, but also in the publication of accurate works on advanced mathematical topics in America.

E. R. HEDRICK.

THE UNIVERSITY OF MISSOURI,
COLUMBIA, MO.,
August, 1905.

TWO BOOKS ON ANALYTIC GEOMETRY.

The Elements of Analytic Geometry. 8vo. 424 pp. *Introduction to Analytic Geometry.* 8vo. 217 pp. By PERCEY F. SMITH and ARTHUR SULLIVAN GALE. New York, Ginn and Company, 1904.

It is a good sign for the university instruction in mathematics of the day that text-books are appearing more frequently in series than formerly. As a rule it means a better individual text-book, for it is the attempt of not one, but several members of the department collaborating to meet in the best possible way what actual experience has taught them to be the need of the student; and it certainly means a more organic, better coördinated mathematical curriculum in the institution which properly uses them. There is furthermore a considerable time economy for the student in the use of such a series. Take for instance the subjects: functions, graphs, partial fractions, and so forth, occurring in different lights in algebra, analytic geometry and calculus; considerable space and time can be gained in their second and subsequent treatments. Still another important gain is to be found in the clearing away for the student of all non-essential difficulties of a new subject, such as becoming accustomed to a new style, a new method of exposition, or a new notation. The books before us are two of a series called into being by needs at Yale University and now appearing under the editorial direction of Professor P. F. Smith.

In those of the works which are at present complete, the main object seems to have been to create drill books, which should be clear and minute in the exposition and analysis of method and rich in exercises, so that a course in one of them would leave the student a ready master of the more usual problems of the subject. If this is a correct conception of their

aim, it has been admirably attained. Each bit of theory is immediately followed by a number of illustrative examples worked out in detail; these are followed by an analysis of the method used in the form of a set of two or three "working rules" for obtaining the required results; then comes a well-stocked and judiciously chosen set of exercises. Some of the special features of the books are well enumerated in the little announcements distributed by the publishers and need not be commented upon here. As the "Introduction" is built upon the first nine chapters of the "Elements" we may perhaps do best to touch first upon the topics common to both books, and then add a few words about the "Elements."

In looking over a new analytic geometry one usually turns straight to the chapter on locus and equation, as the vital part of the whole subject. In these books the chapter does not start by considering indeterminate equations and interpreting them geometrically as a series of points lying on a curve of given type, but proposes and attacks the problem of finding the equation of a given geometric locus at once. The "rule" formulated is fundamental and ought to be in every text book on the subject. An interesting point in this chapter is this: If a given equation the nature of whose locus it is required to discuss is seen to be a special case of a more general equation previously discussed, its locus is a special case of the general locus — this fact is made a principle, and called the "principle of comparison." It is an example of the detail with which the paths of thought are mapped out for the student. Indeed, if the books have any salient fault, it is this spirit of doing so much for the pupil, leaving him to do rather little thinking for himself and reducing nearly everything to mechanical rule.

Although the conic sections occupy a position avowedly secondary to the general methods of analytic geometry, they receive a systematic discussion in a separate chapter. The general equation of the conic is first derived in polar coördinates in terms of eccentricity and distance of focus from directrix. The forms of the special conics are then discussed, first the parabola, then the ellipse and hyperbola in parallel columns. The equations are then reduced to their simplest forms in rectangular coördinates. The more important properties of the conics and systems of conics are considered, and a general method is given for reducing and plotting the locus of the general equation of the second degree.

Tangents and normals receive discussion in a separate chapter, and the analytic geometry of space is treated in the two concluding chapters as a direct generalization of the developments for the plane.

The "Elements," while designed to meet the needs of the same class of students as the shorter book, constitutes a somewhat longer course, and is sufficiently rich in material to allow of considerable latitude of choice. A valuable chapter is added on equations in parametric form, also one on invariants of the quadric with respect to motions of the plane, one on euclidean transformations, one on inversion, one on poles and polars, including polar reciprocation, and the analytic geometry of space receives a much fuller treatment. The chapter on "Line and quadric" treats of tangent lines and planes, diametral and polar planes, and circumscribed and asymptotic cones.

The general appearance of the book is exceedingly attractive. A variety of styles of type has been skillfully used to make clear the different characters of various parts of the text, as theorem, proof, rule, and example. The figures are excellent, clear and distinct, and mention should be made of the half-tones of models of quadric surfaces toward the end of the books, which will do a great deal in helping the student to a correct idea of these surfaces.

No estimate of a book can be quite satisfactory without a basis of class-room experience; but these two books certainly merit a trial, especially in institutions where the needs of the students are similar to those at Yale University.

O. D. KELLOGG.

PRINCETON, N. J.,
July 25, 1905.

NOTES.

THE German mathematical society held its annual meeting, in affiliation with the association of German scientists and physicians, at Meran, Tyrol, September 24–30. Twenty-seven titles were announced on the preliminary programme, including reports "On the introduction of the calculus in the gymnasia," by Professor E. CZUBER; "On partial differential equations of physics," by Professor W. WIEN; "New investigations of Riemann's ζ-functions," by Dr. P. EPSTEIN; "The status of

the arithmetic theory of algebraic numbers," by Professor J. WELLSTEIN. A full report of the meeting will appear in a later number of the BULLETIN.

THE annual meeting of the French association for the advancement of science was held at Cherbourg, August 3 to 8, under the presidency of Professor A. GIARD. Professor M. D'OCAGNE was president of sections I and II, mathematics, mechanics and geodesy. Besides various papers on astronomical subjects, the following mathematical papers were read: "Difficulties in rational mechanics," by Professor P. APPELL; "The forms of functional singularities defined by a vanishing discriminant," by Professor CLARK; "On kinematic methods of determining the radius of curvature," by Professor D'OCAGNE; "Construction of magic squares for the first three degrees," by M. TARRY; "On the number of primes between 1 and n," by Professor LEBON; "Arithmetic spaces," by M. ARNOUX; "On the theory of polygonal numbers," by M. A. CABREIRA. After the meeting various excursions were made to points of interest in the vicinity of Cherbourg.

THE annual list of American doctorates published in *Science* presents for the academic year 1904–1905 324 names, of which 213 are credited to the sciences. The following 21 successful candidates offered mathematics as major subject (the titles of the theses are appended): O. P. AKERS, Cornell, "On the congruence of axes in a bundle of linear complexes"; R. B. ALLEN, Clark, "On hypercomplex number systems belonging to an arbitrary domain of rationality"; C. E. BROWN, Clark, "A study of the simple arithmetic processes"; Miss E. M. CODDINGTON, Columbia, "The historical development of pseudo-spherical surfaces"; W. B. FORD, Harvard, "On the problem of analytic extension as applied to functions defined by power series"; J. S. FRENCH, Clark, "On the theory of the pertingents to a plane curve"; J. N. GATES, Clark, "Cubic and quartic surfaces in four-fold space"; O. E. GLENN, Pennsylvania, "The determination of the abstract groups of order p^2qr, p, q, r being distinct primes"; U. S. HANNA, Pennsylvania, "The bitangentials of the plane quintic and plane sextic"; A. M. HILTEBEITEL, Princeton, "The problem of two fixed centers and certain of its generalizations"; H. E. JORDAN, Chicago, "Group characters of various types of linear groups"; H. G. KEPPEL, Clark, "The cubic three-spread ruled with

planes in four-fold space"; R. B. McCLENON, Yale, "On simple integration with variable limits"; Miss A. M. McKELDON, Pennsylvania, "Groups of order 2^m that contain cyclic subgroups of orders 2^{m-1}, 2^{m-2}, 2^{m-3}"; T. E. McKINNEY, Chicago, "Concerning a certain type of continued fractions depending upon a variable parameter"; W. A. MANNING, Stanford, "Studies on the class of primitive substitution groups"; R. L. MOORE, Chicago, "Sets of metrical hypotheses of geometry"; J. C. MOREHEAD, Yale, "Numbers of the form $2^q - 1$ and Fermat's numbers"; H. B. PHILLIPS, Johns Hopkins, "Some invariants and covariants of ternary collineations"; A. W. SMITH, Chicago, "The symbolic treatment of differential geometry"; R. P. STEPHENS, Johns Hopkins, "I. On a curve of the fifth class, II. On a system of parastroids." The number of American doctorates in mathematics for each year from 1898 to 1905 is 11, 13, 11, 18, 8, 7, 14, 21.

IN the new edition of the Encyclopedia Americana about to be issued by the *Scientific American*, special attention has been given to mathematics. Over forty articles have been written for the encyclopedia by American mathematicians on their several specialties. Professor C. J. KEYSER is in editorial charge of this department of the encyclopedia.

THE publishing house of B. G. Teubner in Leipzig has announced a collection of mathematical models which will henceforth be sold by the firm. The collection at present consists of 59 models, selected from the exhibit prepared by Professor H. WIENER for the third congress of mathematicians at Heidelberg in 1904. It is to be increased and systematized and is intended for service as illustration in the class room rather than for original investigation.

IN its issue of September 15, 1905, the *Enseignement mathématique* begins the publication of the results of its inquiry regarding the method of work of mathematicians, noticed in the BULLETIN, volume 11, page 95. The first installment of eight pages is mainly concerned with answers to question 1*a*, "At what age and under what circumstances did you acquire a taste for mathematics?"

THE various foreign universities named below offer courses in mathematics during the winter semester of 1905–1906 as follows:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic geometry, four hours; Theory of analytic functions, I., four hours; Applications of elliptic functions, 2 hours; Colloquium and seminar, two hours. — By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours; — By Professor F. SCHOTTKY: Theory of abelian functions, four hours; Theory of plane potential, two hours; Seminar, two hours. — By Professor J. KNOBLAUCH: Approximate calculation of definite integrals, one hour; Theory of elliptic functions, four hours; Analytic mechanics, four hours. — By Professor G. HETTNER: Infinite series, products and continued fractions, two hours. — By Professor L. LEHMANN-FILHÉS: Integral calculus, four hours; with exercises, one hour. — By Dr. I. SCHUR: Differential calculus, four hours; with exercises, one hour; Irrational numbers, one hour. — By Dr. E. LANDAU: Algebra, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Differential geometry, four hours; Introduction to the calculus of variations, two hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Elements of analytic geometry of the plane and of space, four hours; with exercises, one hour; General theory of functions, three hours; with exercises, one hour; Seminar, one hour. — By Professor F. LONDON: Differential and integral calculus, II., four hours; with exercises, one hour; Synthetic geometry, two hours; Seminar, one hour.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Elements of determinants, two hours; Modern methods in analytic geometry, three hours; Seminar, one hour. — By Professor R. STURM: Theory of geometric correspondences, II, four hours; Line geometry, two hours; Seminar, two hours. — By Professor A. KNESER: Differential and integral calculus, five hours; Theory of functions, three hours; Seminar, one hour. — By Professor G. LANDSBERG: Algebraic exercises, two hours; Theory of equations, four hours; Introduction to the theory of algebraic functions, two hours.

UNIVERSITY OF FREIBURG. — By Professor J. LUROTH: Plane analytic geometry and differential calculus, five hours; Calculus of variations, three hours; Seminar, one hour. — By Professor L. STICKELBERGER: Theory of differential equations, four hours; Elements of the theory of numbers, two hours. — By Professor P. KÖNIGSBERGER: Elements of par-

tial differential equations, three hours. — By Professor A. LOEWY: Algebraic analysis, four hours; Introduction to the mathematics of insurance, two hours; Seminar, one hour. — By Dr. K. SEITH: Projective geometry, two hours.

UNIVERSITY OF GIESSEN. — By Professor M. PASCH: Introduction to algebraic equations, four hours; Seminar, one hour. —By Professor E. NETTO: Differential and integral calculus, four hours; Analytic geometry of space, four hours; Seminar, two hours.— By Professor H. GRASSMANN: Geometric theory of functions with application to depiction, four hours; Graphical statics, three hours.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ: Theory of potential, four hours; Calculus of variations, two hours; Seminar, two hours. — By Professor F. ENGEL: Analytic mechanics, I, four hours; Algebra, four hours; Differential equations and groups of transformations, two hours, Seminar, two hours. — By Professor K. TH. VAHLEN: Differential calculus, four hours; Theory of probabilities, two hours, with exercises, one hour.

UNIVERSITY OF HEIDELBERG: Professor L. KOENIGSBERGER: Analytic mechanics, four hours; Elliptic functions, two hours; Selected chapters of integral calculus (calculus of variations), two hours; Seminar, two hours. — By Professor E. VALENTINER: Planetary orbits, two hours. — By Professor M. CANTOR: Differential and integral calculus, four hours, with exercises, one hour; Political arithmetic, two hours. — By Professor K. KOEHLER: Analytic geometry of space, three hours. — By Professor K. BOEHM: Introduction to higher mathematics, three hours; Reading of memoirs, one hour.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Theory of surfaces, three hours; Differential equations of one variable, three hours; Seminar, one hour. — By Professor L. HEFFTER (not yet announced): By DR. E. WEINNOLDT: Graphical statics, two hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Differential and integral calculus, four hours; Seminar, one hour. — By Professor A. MAYER; Analytic mechanics, four hours, with exercises, one hour. — By Professor O. HÖLDER: Elliptic functions, four hours; Galois's theory of algebraic equa-

tions, two hours; Seminar, two hours. — By Professor K. ROHN: Analytic geometry of space, five hours, with exercises, one hour; Descriptive geometry, II, two hours, with exercises, two hours; Seminar, two hours. — By Professor F. HAUSDORFF: Theory of transformation groups, three hours. — By Professor H. LIEBMANN: Theory of potential, two hours; Graphical statics, two hours.

UNIVERSITY OF MARBURG. — By Professor K. HENSEL: Algebra, four hours; Theory of surfaces and twisted curves, four hours; Seminar, two hours. — By Professor E. NEUMANN: Theory of functions, four hours; Analytic geometry of space, two hours; Seminar, two hours. — By Dr. F. v. DALWIGK: Statics, two hours; Applied mathematics, two hours; Higher questions of elementary mathematics, one hour. — By Dr. F. JUNG: Integral calculus, five hours.

UNIVERSITY OF ROSTOCK. — By Professor O. STAUDE: Differential and integral calculus, four hours; Theory of curves and surfaces, four hours; Seminar, two hours. — By Professor J. WACHSMUTH: Theory of potential, three hours.

UNIVERSITY OF STRASSBURG. — By Professor TH. REYE: Modern methods in the analytic geometry of space, three hours; Mathematics of elasticity of rigid bodies, two hours; Seminar, two hours. — By Professor H. WEBER: Differential and integral calculus, four hours; Encyclopedia of elementary mathematics, three hours; Seminar, two hours. — By Professor M. SIMON: History of mathematics in antiquity, two hours. — By Professor J. WELLSTEIN: Elliptic integrals, two hours; Determinants and matrices, three hours; Seminar, two hours. — By Professor E. TIMERDING: Plane analytic geometry with exercises, five hours; Graphical statics with exercises, three hours. — By Dr. P. EPSTEIN: Modern investigations in the theory of analytic functions, one hour; Seminar (with Professor Weber), two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. v. BRILL: Introduction to higher mathematics, four hours; Theory of algebraic curves, three hours; Seminar, two hours. — By Professor H. v. STAHL: Higher analysis, II (integral calculus), four hours; Partial differential equations, three hours; Seminar, two hours. — By Professor L. MAURER: Descriptive geometry, II, with exercises, three hours; Elliptic functions, two hours.

UNIVERSITY OF BASEL. — By Professor H. KINKELIN: Differential and integral calculus, I, three hours; Definite integrals, two hours; Partial differential equations, two hours; Algebraic analysis, two hours; Seminar, one hour. — By Professor O. SPEISS: Algebraic equations, two hours; Synthetic geometry, two hours; Elliptic functions, three hours.

UNIVERSITY OF BERN. — By Professor J. H. GRAF: Bessel functions, three hours; Definite integrals and gamma functions, three hours; Differential equations, two hours; Differential and integral calculus, two hours; Theory of functions, two hours; Mathematics of investments and insurance, two hours; Seminar, two hours. — By Professor K. OTT: Integral calculus, two hours; Plane analytic geometry, II, two hours. — By Professor G. HUBER: Celestial mechanics, two hours; Introduction to the theory of algebraic surfaces, three hours.

UNIVERSITY OF GENEVA. — By Professor C. CAILLER: Differential and integral calculus, three hours; Rational mechanics, three hours: Seminar, two hours. — By Professor H. FEHR: Algebra, two hours; Analytic geometry, two hours; Seminar, one hour. — By Dr. J. LYON: Algebraic theory of quadratic forms, one hour. — By Dr. R. DE SAUSSURE: Geometry of motion, two hours; Mechanics of fluids, one hour.

UNIVERSITY OF LAUSANNE. — By Professor M. AMSTEIN: Differential and integral calculus, I, six hours; II, two hours; with exercises, two hours; theory of functions, three hours. By Professor H. JOLY: Descriptive geometry, I, five hours; Analytic geometry, two hours; Synthetic geometry, two hours; Plane curves, two hours. — By Professor O. MAYOR: Rational mechanics, five hours; with exercises, one hour. — By Dr. C. JACCOTTET: Spherical harmonics, two hours.

ACADEMY OF NEUCHATEL. — By Professor L. ISELY: Calculus with geometric applications, three hours; Projective geometry, two hours. — By Professor L. GABEREL: Analytic mechanics, two hours.

UNIVERSITY OF ZÜRICH. — By Professor H. BURKHARDT: Elements of differential and integral calculus, four hours; Theory of potential, four hours; Seminar, two hours. — By Professor A. WEILER: Descriptive geometry, I, with exercises, four hours; Synthetic geometry, three hours; Analytic

geometry, two hours. — By Dr. E. GUBLER: Algebraic analysis with exercises, two hours; Determinants, one hour; Spherical trigonometry, one hour; Geometric pedagogy, one hour.

BEGINNING on Friday, December 1, 1905, Professor V. F. BJERKNES, of the University of Stockholm, will deliver at Columbia University a course of lectures on the subject of "Fields of Force." These lectures are open, without charge, to all teachers and advanced students of physics who may desire to attend. They will be given on Fridays from 4 to 6 P. M. and on Saturdays from 10 to 12 A. M., December 1–23 inclusive. The topics to be treated in the several lectures are enumerated below:

December 1–2.—Elementary investigation of the geometric properties of hydrodynamic fields (with experiments).

December 8–9.—Geometric and dynamic properties of electromagnetic fields according to Maxwell's theory.

December 15–16.—Transformation of the hydrodynamic equations to forms which prove the analogy of hydrodynamic and stationary electromagnetic fields. Further development and discussion of this analogy.

December 22–23.—General conclusions. Remarks on methods of research and of instruction in theoretical physics. Supplementary lecture on the hydrodynamic fields of force in the atmosphere and the sea. Discussion of the fundamental problem of meteorology and hydrography.

The lectures of December 1–2 are introductory and require no special preparation. The most important lecture is that of December 15. Vector analysis as developed by Gibbs will be employed in the course.

PROFESSOR J. H. GRAF, of the University of Bern, and Professor R. v. LILIENTHAL, of the University of Münster, have been elected rectors of their respective institutions for the present academic year.

PROFESSOR R. HAUSSNER, of the technical school of Karlsruhe, has been appointed professor of mathematics at the University of Jena.

PROFESSOR F. SCHUR, of the technical school at Karlsruhe, has declined a call to the technical school at Charlottenburg, as successor to the late Professor Guido Hauck.

PROFESSOR O. TUMLIRZ, of the University of Czernowitz, has been appointed professor of mathematical physics at the University of Innsbruck, as successor to PROFESSOR K. F. J. EXNER, who has retired from active service.

PROFESSOR CH. MÉRAY, of the University of Dijon, has retired from active teaching with the title of honorary professor.

DR. J. BENDIXON has been appointed professor of mathematics at the University of Stockholm.

THE following have been appointed masters of mathematical conferences at the various French universities: ESCLANGON at Bordeaux, DULAC at Grenoble, CLAIRIN at Lille, AUTONNE at Lyon, LEBESGUE at Rennes, BOURGET at Toulouse.

MR. J. H. JEANS, M.A., of Trinity College, Cambridge, England, has been appointed to a professorship of applied mathematics at Princeton University.

AT Columbia University Mr. A. R. MAXSON has been appointed tutor in mathematics.

MR. G. C. DALY, Mr. H. W. POWELL, and Mr. L. P. SICELOFF have been appointed tutors in mathematics in the College of the City of New York.

DR. SAUL EPSTEEN has retired from the editorial staff of the *American Mathematical Monthly*, and has been succeeded by Professor O. E. GLEN, of Drury College.

AT Iowa College, Grinnell, Iowa, Professor S. J. BUCK has retired, having been professor of mathematics at that institution for over forty years. Assistant Professor W. J. RUSK has been promoted to a full professorship of mathematics.

DR. S. KRUSPER, professor emeritus of mathematics and geodesy at the technical school of Budapest, died July 2 at the age of 87 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIERMANN (O.). Vorlesungen über mathematische Näherungsmethoden. Braunschweig, Vieweg, 1905. 8vo. 10 + 227 pp. M. 8.00

BOHLER (O.). Ueber die Picard'schen Gruppen aus dem Zahlen-Körper der dritten und vierten Einheitswurzel. Zürich, 1905. 8vo. 102 pp. M. 3.00

BUCHERER (A. H.). Elemente der Vektor-Analysis. Mit Beispielen aus der theoretischen Physik. 2te Auflage. Leipzig, Teubner, 1905. 8vo. 8 + 103 pp. Cloth. M. 2.40

DALHUISEN (A. A.). Over eenige aantallen van kegelsneden, die aan acht voorwaarden voldoen. Utrecht, 1905. 8vo. 8 + 86 pp. M. 3.00

FESTSCHRIFT Adolf Wüllner gewidmet zum siebzigsten Geburtstage, 13. Juni 1905, von der Königl. Technischen Hochschule zu Aachen, ihren früheren und jetzigen Mitgliedern. Leipzig, Teubner, 1905. 8vo. 8 + 264 pp., 1 portrait, 8 plates. M. 8.00

FIEDLER (W.). Analytische Geometrie der Kegelschnitte mit besonderer Berücksichtigung der neueren Methoden. Nach George Salmon frei bearbeitet. 6te Auflage. Zweiter Teil. Leipzig, Teubner, 1905. 8vo. 24 + 412 pp. M. 8.00

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. 4te Auflage. Braunschweig, Vieweg, 1905. 8vo. 15 + 217 pp. M. 5.00

GEER (P. VAN). De herleving der mathematische wetenschappen. Leiden, 1904. 8vo. 45 pp. M. 1.00

GOURSAT (E.). Cours d'analyse mathématique. Vol. II: Fonctions analytiques; équations différentielles; équations aux dérivées partielles; éléments du calcul des variations. (En 3 fascicules.) Fascicule 2: Pp. 305–528. Paris, Gauthier-Villars, 1905. 8vo. (Cours de la Faculté des sciences de Paris.)

GROTENDORST (N. C.). Beginselen der differentiaal- en integraalrekening. 2te, verbederde druk. Breda, 1904. 8vo. 10 + 514 pp. M. 13.00

GULDBERG (A.). Sur les équations linéaires aux différences finies. 4to. (*Annales scientifiques de l'École normale supérieure* 21 (1905), pp. 309–348.)

HAGEN (J. G.). Synopsis der höheren Mathematik. Vol. III: Differential- und Integralrechnung. 6te Lieferung: Pp. 321–384. Berlin, Dames, 1905. 4to. M. 5.00

HILL (G. W.). The collected mathematical works. Vol. I. Published by the Carnegie Institution of Washington, June, 1905. 4to. [With an introduction by H. Poincaré.] 18 + 363 pp., portrait. Cloth.

KOEBE (P.). Ueber diejenigen analytischen Funktionen eines Arguments, welche ein algebraisches Additionstheorem haben. Berlin, 1905. 8vo. 32 pp. M. 1.50

LÁSKA (W.). Wyklady nomografii. Opracowal i wydal F. Ulkowski. Lwów, 1904–1905. (autogr.)

OFFERHAUS (H.). Lineaire kegelsneestelses en weefsels. Groningen, 1905. 8vo. 8 + 100 pp., 2 plates. M. 3.00

PICARD (E.). Traité d'analyse. 2e édition, revue et augmentée. Vol. II: Fonctions harmoniques et fonctions analytiques; introduction à la théorie des équations différentielles; intégrales abéliennes et surfaces de Riemann. Paris, Gauthier-Villars, 1905. 8vo. 15 + 587 pp. Fr. 18.00

QUITMANN (E.). Ueber Minimallagen in ebenen Gebieten. Münster, 1905. 8vo. 45 pp., 1 plate. M. 1.50

RULAND (N.). Praktische Anleitung zum gründlichen Unterrichte in der höheren Mathematik. Ausführliche Auflösung der in Dr. Ed. Heis' Sammlung von Beispielen u. s. w. enthaltenen Aufgaben. Teil 3: Kettenbrüche, Teilbruchreihen, Permutationen, Kombinationen, Variationen, Wahrscheinlichkeitsrechnung, binomischer und polynomischer Lehrsatz, figurierte Zahlen, Gleichungen von höheren Graden, transzendente Gleichungen, Anwendung der Algebra auf Aufgaben aus der Geometrie, Physik und Astronomie. Zum Selbstunterrichte bestimmt. 2te, verbesserte Auflage. Bonn, Cohen, 1905. 8vo. 8 + 491 pp. M. 6.00

RUTGERS (J. G.). Over differentialen van gebroken orde en haar gebruik bij de afleiding van bepaalde integralen. Utrecht, 1904. 8vo. 56 pp. M. 2.50

SALMON (G.). See FIEDLER (W.).

SCHOUTE (P. H.). Mehrdimensionale Geometrie. Teil 2: Die Polytope. Leipzig, Göschen, 1905. 12mo. Cloth. (Sammlung Schubert 36.) M. 10.00

SCHUBERT (H.) Auslese aus meiner Unterrichts- und Vorlesungspraxis. Vol. I. Leipzig, Göschen, 1905. 12mo. M. 4.00

TANNERY (J.). Introduction à la théorie des fonctions d'une variable. 2e édition, entièrement refondue. Vol. I: Nombres irrationnels; ensembles; limites; séries; produits; infinis: fonctions élémentaires; dérivées. Paris, Hermann, 1904. 8vo. 9 + 423 pp. Fr. 14.00

ULKOWSKI (F.). See LÁSKA (W.).

ZAALBERG (A. L.). Differentiaal-meetkundige eigenschappen van stralenstelsels. Leiden, 1905. 8vo. 8 + 120 pp. M. 2.50

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). First algebra. London, Bell, 1905. 12mo. 186 pp. Cloth. 1s. 6d.

—— With answers. 2s.

—— Key to elementary algebra. London, Bell, 1905. 12mo. Cloth. 10s.

BENOIT (P.). Essai d'une géométrie nouvelle. Saint-Dié, Weick, [1905]. 8vo. 159 pp.

BIDSCHOF (F.) und VITAL (A.). Fünfstellige mathematische und astronomische Tafeln. Zum Gebrauche für Mathematiker, Astronomen, Geographen und Seeleute zusammengestellt und mit Formelsammlungen versehen. Wien, 1905. 8vo. Cloth. M. 7.50

BÖTTCHER (R.) und SENDLER (R.). Raumlehre für Lehrerbildungsanstalten. Teil 2: Stereometrie und Trigonometrie. Resultate. Breslau, Handel, 1905. 8vo. 15 pp. M. 0.25

BORCHARDT (W. G.). Elementary algebra. With answers. London. Rivington, 1905. 12mo. 564 pp. Cloth. 4s. 6d.

BOURNE (A. A.). See BAKER (W. M.).

BRIDGETT (R. C.). An introduction to algebra. London, Blackie, 1905. 12mo. 96 pp. Cloth. 1s.

BUSH (W. N.) and CLARKE (J. B.). The elements of geometry. New York, Silver, Burdette, & Co., [1905]. 12mo. 12 + 355 pp. Half leather. $1.25

CLARKE (J. B.). See BUSH (W. N.).

DIEKMANN (J.). See KOPPE.

GÉOMÉTRIE. Cours moyen. Par une réunion de professeurs. Paris, Poussielgue, [1905]. 16mo. 191 pp. (Collection d'ouvrages classiques rédigés en cours gradués.)

GILDEMEISTER (T.). The multiplication tables. Chicago, Flanagan, [1905]. 12mo. 31 pp. $0.10

GRÉVY (A.). Trigonométrie, à l'usage des élèves des classes de seconde et première C et D et mathématiques A et B (programmes du 31 mai 1902). 3e édition. Paris, Vuibert et Nony, 1905. 16mo. 8 + 274 pp.

KOPPE und DIEKMANN. Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Zweiter Teil der Planimetrie, Stereometrie und Trigonometrie. Ausgabe für Reallehranstalten. (19te Auflage.) 3te Auflage der neuen Bearbeitung von J. Diekmann. Mit zahlreichen Uebungen und Aufgaben. Essen, Baedeker, 1905. 8vo. 4 + 270 pp. Cloth. M. 2.40

LANNER (A.). Die wissenschaftlichen Grundlagen des ersten Rechenunterrichts. Wien und Leipzig, Fromme, 1905. K. 1.20

LAYNG (A. E.). A text-book of algebra; embodying graphic and other solutions and applications to arithmetic and geometry. Part I. With answers. London, Blackie, 1905. 12mo. 206 pp. Cloth. 2s. 6d.

LIASSE. Démonstration du postulatum d'Euclide. Boulogne-sur-Mer, Hamain, 1905. 8vo. 8 pp.

LIGHTFOOT (J.). Graphic algebra. For elementary and intermediate students. With exercises and answers. London, Ralph, 1905. 12mo. 92 pp. Cloth. 1s.

LÓPEZ (D. L.). Soluciones razonadas á las problemas de geometría que exige el programa de oposiciones al Cuerpo de telégrafos. Madrid, Alonso, 1905. 8vo. 19 pp. Fr. 1.00

MORAWETZ (J.). Ueber die Grundlagen der Arithmetik und Algebra im Mittelschulunterrichte. Salzburg, 1904. 8vo. 34 pp. M. 1.50

MÜLLER (O.). Tavole di logaritmi con cinque decimali. 8a edizione, aumentata delle tavole dei logaritmi d'addizione e sottrazione, per cura di M. Rajna. Milano, Hoepli, 1905. 16mo. 36 + 191 pp. (Manuali Hoepli.)

NOUVELLES tables de logarithmes à cinq décimales pour les lignes trigonométriques dans les deux systèmes de la division centésimale et de la division sexagésimale du quadrant, et pour les nombres de 1 à 12,000, suivies des mêmes tables à quatre décimales et de diverses tables et formules usuelles. 2e édition, revue et corrigée. Paris, Gauthier-Villars, 1905. 8vo. (Service géographique de l'armée.)

PILLET (F.). See ROSSIGNON (A.).

RAJNA (M.). See MÜLLER (O.).

ROSSIGNON (A.) et PILLET (F.). Eléments de géométrie, à l'usage de l'enseignement secondaire, de l'enseignement primaire supérieur et des écoles normales, accompagné de notions sur l'arpentage, le levé des plans et le nivellement. Paris, Rudeval, 1905. 8vo. 512 pp. Fr. 5.50

SCHROETER (R.). Die Methodik des Rechen- und Raumlehre-Unterrichts in der Volksschule. Ein Handbuch für die oberen Klassen der Seminare und für Volksschullehrer. 3te, vollständig umgearbeitete und verbesserte Auflage. Wittenberg, Herrosé, 1905. 8vo. 7 + 372 pp. M. 3.50

SENDLER (R.). See BÖTTCHER (R.).

VITAL (A.). See BIDSCHOF (F.).

WISKUNDIGE OPGAVEN met de oplossingen, door de leden van het wiskundig genootschap. Deel X, stuk 2. Amsterdam, 1904. 8vo. Pp. 81-160, 1 plate.

III. APPLIED MATHEMATICS.

APPELL (P.). Cours de mécanique, à l'usage des élèves de la classe de mathématiques spéciales, conforme au programme du 27 juillet 1904. 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1905. 8vo. 498 pp. Fr. 12.00

BERNHARD (M.). Darstellende Geometrie mit Einschluss der Schattenkonstruktionen und der Perspektive. Als Leitfaden für den Unterricht an technischen Lehranstalten, Oberrealschulen und Realgymnasien, sowie zum Selbststudium herausgegeben. 2te, verbesserte und stark vermehrte Auflage. Stuttgart, Enderlen, 1905. 8vo. 11 + 278 pp. M. 5.20

BOURGUIGNON (P.). Cours de cinématique théorique et appliquée. Vol. I: Cinématique théorique, à l'usage des élèves des écoles des arts et metiers. Paris, Paulin, 1905. 8vo. 185 pp. Fr. 5.00

BOURGONNIER (A.) et ROLLET (P.) Cours de mécanique élémentaire. Vol. I: Cinématique, à l'usage des élèves des classes de première C et D. 7 + 201 pp. Vol. II: Statique et dynamique (classes de mathématiques A et B; préparation aux écoles navales et de Saint-Cyr). 288 pp. Paris, Paulin, 1904, 1905. 8vo. Fr. 3.00; Fr. 2.50

CARLIER (J. G.). Les méthodes et appareils de mesure du temps, des distances, des vitesses et des accélérations. (En 2 volumes.) Vol. II. Paris, 1905. 8vo. Fr. 6.00

DUFF (A. W.). Elementary experimental mechanics: Kinematics, dynamics, elastic solids and fluids. New York, Macmillan, 1905. 12mo. 7 + 267 pp. Cloth. $1.60

EREDE (G.). Manuale di geometria pratica. 4ta edizione, riveduta ed aumentata. Milano, Hoepli, 1905. 16mo. 16 + 257 pp. (Manuali Hoepli.) Fr. 2.00

FISH (J. C. L.). Mathematics of the paper location of a railroad. New York, Engineering News Publishing Company, 1905. 8vo. 18 pp. $0.25

FÖPPL (A.). Vorlesungen über technische Mechanik. Vol. I: Einführung in die Mechanik. 3te Auflage. Leipzig, Teubner, 1905. 8vo. 16 + 428 pp. Cloth. M. 10.00

GALILEI (G.). Le opere. Edizione nazionale sotto gli auspizi di Sua Maestà il Re d'Italia. Vol. XVI: 1. Carteggio 1634–1636. 2. Indice cronologico delle lettere contenute nel Vol. XVI. 3. Indice alfabetico delle lettere contenute nel Vol. XVI. Firenze, Barbèra, 1905. 4to. 564 pp.

GEITLER (J. von). Elektromagnetische Schwingungen und Wellen. Braunschweig, Vieweg, 1905. 8vo. 8+154 pp. (*Die Wissenschaft*, Sammlung naturwissenschaftlicher und mathematischer Monographien, Vol. 6.) M. 4.50

GIBBS (J. W.). Elementare Grundlagen der statistischen Mechanik entwickelt besonders im Hinblick auf eine rationelle Begründung der Thermodynamik. Deutsch bearbeitet von E. Zermelo. Leipzig, Barth, 1905. 8vo. 16+216 pp. M. 10.00

GLEICHEN (A.). Vorlesungen über photographische Optik. Leipzig, Göschen, 1905. 12mo. M. 9.00

GUICHARD (C.). Traité de mécanique. Première partie: Cinématique, à l'usage des élèves des classes de première C et D. 3e édition. Paris, Vuibert et Nony, 1905. 8vo. 8+108 pp. Fr. 1.50

HENDRICKSON (W. W.). Notes on stereographic projection and the astronomical triangle. Annapolis, Md., United States Naval Institute, 1905. 16mo. 31 pp. Cloth. (Sold to midshipmen only.)

HEYDEMAN (W. J.). Beginselen der beschrijvende meetkunde. Amsterdam, Ahrend & Zoon, 1905. 8vo. Cloth. Fl. 5.60

LEA (S. H.). Hydrographic Surveying. Methods, tables, and forms of notes. New York, Engineering News Publishing Company, 1905. 8vo. 200 pp. Cloth. $2.00

LORD (H. C.). The elements of geodetic astronomy, for civil engineers. Columbus, O., Lord, [1905]. 8vo. 150 pp. Cloth. $1.50

MATRICULATION model answers: Mechanics. From June 1890 to June 1898 and from September 1902 to June 1905. London, Clive, 1905. 12mo. 186 pp. (University tutorial series.) 2s.

PATON (J.). Petit traité mathématique et pratique des opérations commerciales et financières. Vol. II: Eléments d'algèbre financière. Paris, Vuibert et Nony, 1905. 8vo. 424 pp. (Bibliothèque d'enseignement technique.)

ROLLET (P.). See BOURGONNIER (A.).

SACHS (L.). Zur Berechnung räumlicher Fachwerke. Allgemeine Formeln für statisch bestimmte und insbesondere statisch unbestimmte Kuppel-, Zelt- und Turmdächer. Berlin, Ernst, 1905. 8vo. 4+56 pp., 3 plates. M. 2.50

SCHÜTTE (F.). Anfangsgründe der darstellenden Geometrie für Gymnasien. Düren, 1905. 8vo. 42 pp. M. 1.50

SIERTSEMA (L. H.). De electriciteitsgeleiding in gassen, in verband met de electronentheorie. Delft, 1904. 8vo. 22 pp. M. 1.00

THOMSON (J. J.). Elektrizitätsdurchgang in Gasen. Deutsche autorisierte Ausgabe unter Mitwirkung des Autors besorgt und ergänzt von E. Marx. (In 3 Lieferungen.) Lieferung I. Leipzig, Teubner, 1905. 8vo. 217 pp. M. 6.00

THULOUP (A.). La sphéricité de la terre; sa rotation et le pendule de Foucault. Melun, Imprimerie administrative, 1905. 8vo. 31 pp. (Ministère de l'instruction publique et des beaux-arts. Musée pédagogique, service des projections lumineuses. Notice sur les vues.)

TIDDENS (P. G.). Beschwouwingen over den loop der lichtstralen en de beeldvorming in optische stelsels. Leiden, 1904. 8vo. 98 pp., 1 plate.

VINTÉJOUX (F.). Nouvelles tables d'intérêts composés et d'annuités et précis de la théorie et pratique des opérations financières à long terme, particulièrement des emprunts. Paris, 1905. 8vo. 12 + 226 pp. Fr. 10.00

WETHAM (W. C. D.). The theory of experimental electricity. London and New York, Macmillan, 1905. 8vo. 11 + 334 pp. Cloth. $2.50

ZERMELO (E.). See GIBBS (J. W.).

ZEUNER (G.). Théorie des turbines. Traduit de l'allemand par E. Kreitmann. Paris, Dunod et Pinat, 1905. 8 vo. (Hydraulique pratique.) Fr. 14.00